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# The gluing problem in the fusion systems of the symmetric, alternating and linear groups.

Assaf Libman

*Department of Mathematical Sciences, King's College, University of Aberdeen, Aberdeen AB24 3UE , Scotland, U.K.*

**Abstract**

A solution to Linckelmann's gluing problem in all fusion systems of blocks yields an alternative formulation of Alperin's weight conjecture. Using Bredon equivariant cohomology techniques we solve the gluing problem in fusion systems (of blocks) which are isomorphic to the fusion systems of the symmetric groups or the alternating groups or $\mathrm{GL}_d(q)$ or $\mathrm{SL}_d(q)$ at the defining characteristic.

*Keywords:* Equivariant homotopy theory, gluing problem, Alperin's conjecture, radical subgroups, permutation groups
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## 1. Introduction

Fix a prime $p$ and an algebraically closed field $k$ of characteristic $p$. Let $\mathcal{F}$ be a saturated fusion system on a $p$-group $S$ and $\mathcal{F}^c$ the full subcategory of the $\mathcal{F}$-centric subgroups, see [7]. An $s$-simplex in $\mathcal{F}^c$ is a sequence $P_0 \lneq \cdots \lneq P_s$ of $\mathcal{F}$-centric subgroups of $S$. It is $\mathcal{F}$-conjugate to $P'_0 \lneq \cdots \lneq P'_s$ if there exists an $\mathcal{F}$-isomorphism $P_s \to P'_s$ which carries every $P_i$ onto $P'_i$. The equivalence classes of $s$-simplices in $\mathcal{F}^c$ form a poset $[S(\mathcal{F}^c)]$ where $[Q_\bullet] \to [P_\bullet]$ if $Q_\bullet$ is $\mathcal{F}$-conjugate to a subsimplex, namely a subsequence or a "face", of $P_\bullet$. For every $i \geq 0$ there is a functor

$$\mathcal{A}^i_{\mathcal{F}} \colon [S(\mathcal{F}^c)] \to \mathbf{Ab}, \qquad [P_\bullet] \mapsto H^i(\mathrm{Aut}_{\mathcal{F}}(P_\bullet); k^\times)$$

where $\mathrm{Aut}_{\mathcal{F}}(P_\bullet)$ is the subgroup of $\mathrm{Aut}_{\mathcal{F}}(P_s)$ that normalises $P_\bullet$ and $k^\times$ is the group of the invertible elements in $k$. See [19] and [17, §1] for more details. A family of classes $\alpha_{[P_\bullet]} \in \mathcal{A}^i_{\mathcal{F}}([P_\bullet])$ is called *compatible* if every $[Q_\bullet] \to [P_\bullet]$ in $[S(\mathcal{F}^c)]$ carries $\alpha_{[Q_\bullet]}$ to $\alpha_{[P_\bullet]}$. In other words, it is an element in $\varprojlim_{[S(\mathcal{F}^c)]} \mathcal{A}^i_{\mathcal{F}}$.

Set $H^i(\mathcal{D}; A) := \varprojlim^i_{\mathcal{D}} A$ where $A \colon \mathcal{D} \to \mathbf{Ab}$ is a functor. The inclusions

*Email address:* a.libman@abdn.ac.uk (Assaf Libman)

$\mathrm{Aut}_{\mathcal{F}}(P_\bullet) \leq \mathrm{Aut}_{\mathcal{F}}(P_s) \subseteq \mathcal{F}^c$ yield a natural map

$$H^i(\mathcal{F}^c; k^\times) \xrightarrow{\ \gamma\ } \varprojlim_{[S(\mathcal{F}^c)]} \mathcal{A}^i_{\mathcal{F}}$$

We say that the *gluing problem* for the family $(\alpha_{[P_\bullet]})$ has a solution if it is in the image of $\gamma$. See [19], [18] and [17, §4] which are the starting point of this paper.

Suppose that $\mathcal{F}$ is the fusion system of a block $b$. Külshammer and Puig's construction in [15, 1.8, 1.12], together with Dade's splitting theorem, yield a compatible family of classes $(\alpha_{[P_\bullet]})$ in $\mathcal{A}^2_{\mathcal{F}}$. Linckelmann showed in [18, 4.2], see also [17, §4], that a solution to the gluing problem for this family of classes in all the fusion systems of blocks yields an alternative formulation to Alperin's weight conjecture.

Little is known about the solution of the gluing problem. By [23, Thms. 1.2, 1.3] it is not unique in general and in contrast, the gluing problem is trivial in the fusion systems of tame blocks. In this paper we will prove the following result.

**Theorem 1.1.** *The gluing problem for any family of compatible classes in* $\mathcal{A}^2_{\mathcal{F}}$ *has a solution in the fusion systems of the symmetric groups and the alternating groups and also* $\mathrm{GL}_d(q)$ *and* $\mathrm{SL}_d(q)$ *at the defining characteristic.*

A *collection* $\mathcal{C}$ in $\mathcal{F}^c$ is a set of $\mathcal{F}$-centric subgroups of $S$ which is closed to $\mathcal{F}$-conjugacy. We let $\mathcal{F}^{\mathcal{C}}$ denote the full subcategory generated by the object set $\mathcal{C}$. A compatible family $(\alpha_{[P_\bullet]})$ in $\mathcal{F}^{\mathcal{C}}$ is an element of $\varprojlim_{[S(\mathcal{F}^{\mathcal{C}})]} \mathcal{A}^i_{\mathcal{F}}$. The gluing problem has a solution in $\mathcal{F}^{\mathcal{C}}$ if $(\alpha_{[P_\bullet]})$ is in the image of $H^i(\mathcal{F}^{\mathcal{C}}; k^\times) \to \varprojlim_{[S(\mathcal{F}^{\mathcal{C}})]} \mathcal{A}^i_{\mathcal{F}}$.

Assume that $\mathcal{F}$ has an associated centric linking system $\mathcal{L}$, see [7]. Let $\mathcal{L}^{\mathcal{C}}$ be the full subcategory of $\mathcal{L}$ with object set $\mathcal{C}$. By [16, Theorem A] there is a functor

$$\mathcal{A}^i_{\mathcal{L}} \colon [S(\mathcal{F}^{\mathcal{C}})] \to \mathbf{Ab}, \qquad [P_\bullet] \mapsto H^i(\mathrm{Aut}_{\mathcal{L}}(P_\bullet); k^\times)$$

where $\mathrm{Aut}_{\mathcal{L}}(P_\bullet)$ is the preimage of $\mathrm{Aut}_{\mathcal{F}}(P_\bullet)$ under the projection $\mathrm{Aut}_{\mathcal{L}}(P_s) \xrightarrow{\pi} \mathrm{Aut}_{\mathcal{F}}(P_s)$. By abuse of notation we denote the image of classes $\alpha_{[P_\bullet]} \in \mathcal{A}^i_{\mathcal{F}}$ under $\pi^*$ by the same letter $\alpha_{[P_\bullet]}$ and may seek for a solution to the gluing problem in $\mathcal{L}^{\mathcal{C}}$. Namely, we want to find a class $\alpha \in H^i(\mathcal{L}^{\mathcal{C}}; k^\times)$ which restricts to $\alpha_{[P_\bullet]} \in \mathcal{A}^i_{\mathcal{L}}([P_\bullet])$ for all $[P_\bullet] \in [S(\mathcal{F}^{\mathcal{C}})]$.

The upshot of the next result is that in the presence of a linking system, the gluing problem can be reduced to any collection $\mathcal{C}$ which contains the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups, [7, Def. A.9]. This is the equivalence of (a) and (d) of the theorem below. It will be proven in §3.

**Theorem 1.2.** *Consider a saturated fusion system* $\mathcal{F}$ *on* $S$ *and assume that it has an associated centric linking system* $\mathcal{L}$*. Assume that* $\alpha_{[P_\bullet]} \in \mathcal{A}^2_{\mathcal{F}}([P_\bullet])$ *is a family of compatible classes in* $\mathcal{F}^c$*. Let* $\mathcal{C}$ *be a collection in* $\mathcal{F}^c$ *which contains all the* $\mathcal{F}$*-centric* $\mathcal{F}$*-radical subgroups. Then the following are equivalent*

(a) *The gluing problem for the family* $(\alpha_{[P_\bullet]})$ *has a solution in* $\mathcal{F}^c$.

(b) *The gluing problem for the family* $(\alpha_{[P_\bullet]})$ *has a solution in* $\mathcal{L}$.

(c) *The gluing problem for the family* $(\alpha_{[P_\bullet]})_{[P_\bullet]\in[S(\mathcal{F}^c)]}$ *has a solution in* $\mathcal{L}^{\mathcal{C}}$.

(d) *The gluing problem for the family* $(\alpha_{[P_\bullet]})_{[P_\bullet]\in[S(\mathcal{F}^c)]}$ *has a solution in* $\mathcal{F}^{\mathcal{C}}$.

*In addition, if* $H^2([S(\mathcal{F}^{\mathcal{C}})];\mathcal{A}^1_{\mathcal{L}})=0$ *then there is a solution to the gluing problem for* any *collection of compatible classes* $\alpha_{[P_\bullet]}\in\mathcal{A}^2_{\mathcal{F}^c}([P_\bullet])$ *in* $\mathcal{F}^{\mathcal{C}}$.

Theorem 1.1 follows from the second part of Theorem 1.2 together with Corollary 3.4 which translates the calculation of higher limit groups into a problem in equivariant homotopy theory. This is an essential step for the fusion systems of the symmetric and alternating groups. For the general linear groups the second statement of Theorem 1.2 is proven directly.

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## 2. Bredon cohomology

Throughout this paper the word “space” is a synonym to “simplicial set”. A $G$-space is simply a simplicial object in the category of $G$-sets. The set of points of a $G$-set $X$ fixed by $H\le G$ is denoted $X^H$. The orbits under the action of $G$ are denoted $X/G$ or $X_G$. The isotropy group of a simplex $x\in X$ is denoted $\mathrm{Iso}_G(x)$.

Let $G$ be a finite group and $\mathcal{O}_G$ the category of transitive $G$-sets; it is equivalent to the full subcategory spanned by the coset spaces $G/H$. A *coefficient functor* $\mathcal{M}$ for $G$ is a contravariant functor $\{G\text{-sets}\}\to\mathbf{Ab}$ which turns coproducts of $G$-sets into products of abelian groups. Equivalently, it is an object in the category $\mathcal{O}_G$-mod of the contravariant functors $\mathcal{O}_G\to\mathbf{Ab}$. See [4, §I-4]. The prime examples for this paper are defined next.

**Definition 2.1.** Fix a finite group $G$, an integer $i\ge 0$ and a $G$-module $M$. Let $\mathbb{Z}[G]$ denote the group ring of $G$. For every $G$-set $X$ let $\mathbb{Z}[X]$ denote the permutation $G$-module $\oplus_{x\in X}\mathbb{Z}$. Define a coefficient functor $\mathcal{H}^i_G(M)$ for $G$ by the assignment

$$\mathcal{H}^i_G(M)\colon X\mapsto \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}[X],M),\qquad \text{where } X \text{ is a } G\text{-set}.$$

If $A$ is an abelian group we view it as a trivial $G$-module and write $\mathcal{H}^i_G(A)$ for this coefficient functor.

Observe that $\mathcal{H}^0_G(A)$ is the constant coefficient functor $G/K\mapsto A$. Shapiro's lemma implies that $\mathcal{H}^1_G(A)\colon G/K\mapsto \mathrm{Hom}(K,A)$, see Proposition B.1(1).

By applying a coefficient functor $\mathcal{M}$ to the sets of simplices of a $G$-space $X$, one obtains a cosimplicial abelian group $\mathcal{M}(X)$. The associated (normalised) cochain complexes are denoted

$$C^*_G(X;\mathcal{M})\qquad\text{and}\qquad NC^*_G(X;\mathcal{M}).$$

They are chain equivalent and their homology groups are called the *Bredon cohomology groups* $H^*_G(X;\mathcal{M})$. Note that $C^n_G(X;\mathcal{M}) = \prod_{[\sigma]} \mathcal{M}([\sigma])$ where $[\sigma]$ runs through all the orbits of the $n$-simplices of $X$. In $NC^n_G(X;\mathcal{M})$ only non-degenerate $\sigma$ appear in the product.

Given a $G$-space $X$, the assignment $G/H \mapsto H_j(X^H;\mathbb{Z})$ defines an object $\underline{H}_j(X;\mathbb{Z})$ in $\mathcal{O}_G$-mod. There is a spectral sequence, [4, Chap. I, (10.4)],

$$E_2^{i,j} = \operatorname{Ext}^i_{\mathcal{O}_G\text{-mod}}(\underline{H}_j(X;\mathbb{Z}),\mathcal{M}) \;\Rightarrow\; H^{i+j}_G(X;\mathcal{M}),$$

which is natural in $X$ and $\mathcal{M}$. It follows that $H^*_G(-;\mathcal{M})$ is homotopy invariant because $X \to Y$ is a $G$-equivalence if and only if $X^H \to Y^H$ are homotopy equivalences. Together with Künneth's theorem, we obtain the following result.

**Lemma 2.2.** *For any $G$-space $X$ and any integral homology isomorphism of spaces $f\colon A \to B$, the map $A \times X \xrightarrow{f\times X} B \times X$ induces an $H^*_G(-;\mathcal{M})$-isomorphism for any coefficient functor $\mathcal{M}$ for $G$.*

A *relative $G$-space* $(X, X')$ is a pair of $G$-spaces $X' \subseteq X$. A $G$-map $f\colon X \to Y$ is a *relative isomorphism* $(X,X') \xrightarrow{\cong} (Y,Y')$ if $f(X') \subseteq Y'$ and if $f$ induces a bijection $(X - X') \xrightarrow{\cong} (Y - Y')$. Whenever we talk about a simplex in the relative space $(X,X')$ we mean a simplex in $X - X'$.

A relative $G$-space $(X,Y)$ gives rise to a short exact sequence

$$0 \to C^*_G(X,Y;\mathcal{M}) \to C^*_G(X;\mathcal{M}) \to C^*_G(Y;\mathcal{M}) \to 0$$

and to the usual long exact sequence in cohomology

$$\cdots \to H^i_G(X,Y;\mathcal{M}) \to H^i_G(X;\mathcal{M}) \to H^i(Y;\mathcal{M}) \to H^{i+1}_G(X,Y;\mathcal{M}) \to \dots$$

The following results are clear from the definitions.

**Proposition 2.3.** *Assume that $Y \subseteq X$ is an inclusion of $G$-spaces and that $\underline{A}$ is the constant coefficient functor $G/K \mapsto A$ for some abelian group $A$. Then $H^*_G(X,Y;\underline{A}) = H^*(\bar{X},\bar{Y};A)$ where $\bar{X}$ and $\bar{Y}$ are the orbit spaces of $X$ and $Y$.*

**Proposition 2.4.** *If $f\colon (X,X') \to (Y,Y')$ is a relative isomorphism of relative $G$-spaces then $H^*_G(Y,Y';\mathcal{M}) \cong H^*_G(X,X';\mathcal{M})$ for any coefficient functor $\mathcal{M}$ for $G$*

Suppose that $K \leq G$. Let $X{\downarrow}^G_K$ denote the restriction of the action of a $G$ on a set $X$ to $K$. For a $K$-set $X$ let $X{\uparrow}^G_K$ denote the induced set $G \times_K X$, namely the orbit space of $G \times X$ under the action of $K$ via $k \cdot (g,x) = (gk^{-1}, kx)$. The functor $X \mapsto X{\uparrow}^G_K$ is left adjoint to the restriction functor $X \mapsto X{\downarrow}^G_K$.

It is straightforward to check that if $K \leq G$ then $(K/I){\uparrow}^G_K = G/I$ for any $I \leq K$. As an immediate consequence we obtain the following result.

**Proposition 2.5.** *Let $(X,X')$ be a relative $G$ space and assume that $Y' \subseteq Y$ are $K$-subspaces of $X$ for some subgroup $K$ where $Y' \subseteq X'$. Then the inclusion $(Y,Y') \subseteq (X,X')$ induces a relative isomorphism $(Y,Y'){\uparrow}^G_K \cong (X,X')$ if and only if it induces a bijection $(Y - Y')/K \cong (X - X')/G$ and $\operatorname{Iso}_G(y) \leq K$ for any $y \in Y - Y'$.*

Restriction and induction are useful in creating new coefficient functors from old.

**Definition 2.6.** Assume that $K \le G$ and that $\mathcal{M}$ is a coefficient functor for $G$. Define a coefficient functor $\mathcal{M}\downarrow_K^G$ for $K$ by the assignment $\mathcal{M}\downarrow_K^G \colon X \mapsto \mathcal{M}(X\uparrow_K^G)$.

If $\mathcal{N}$ is a coefficient functor for $K$ define a coefficient functor $\mathcal{N}\uparrow_K^G$ for $G$ by the assignment $\mathcal{N}\uparrow_K^G \colon X \mapsto \mathcal{N}(X\downarrow_K^G)$.

**Proposition 2.7.** *Fix $K \le G$. Then for any $G$-module $M$ there is an isomorphism $\mathcal{H}^i_G(M)\downarrow_K^G \cong \mathcal{H}^i_K(M\downarrow_K^G)$.*

*Proof.* If $M \to I^\bullet$ is an injective resolution then $I^\bullet\downarrow_K^G = \mathrm{Hom}_G(\mathbb{Z}[G], I^\bullet)$ is an injective resolution for $M\downarrow_K^G$ because $\mathbb{Z}[G]$ is a free $K$-module, hence flat. Therefore, if $X$ is a $K$-set then $\mathrm{Ext}^i_G(\mathbb{Z}[X\uparrow_K^G], M) \cong \mathrm{Ext}^i_G(\mathbb{Z}[G] \otimes_{\mathbb{Z}[K]} \mathbb{Z}[X], M) \cong \mathrm{Ext}^i_K(\mathbb{Z}[X], M\downarrow_K^G)$. □

**Definition 2.8.** Let $\mathcal{M}_1, \dots, \mathcal{M}_n$ be coefficient functors for $G_1, \dots, G_n$. Set $G = G_1 \times \dots \times G_n$ and let $G_{(i)}$ denote the kernel of the projection $G \to G_i$. For any $G$-set $X$ let $X_{(i)}$ denote the $G_i$-set $X/G_{(i)}$. Define a coefficient functor for $G$

$$(\mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_n) : X \ \mapsto \ \oplus_{i=1}^n \mathcal{M}_i(X_{(i)}).$$

**Proposition 2.9.** *With the notation above there is an isomorphism*

$$H^*_{G_1\times G_2}(X; \mathcal{M}_1 \oplus \mathcal{M}_2) = H^*_{G_1}(X/G_2; \mathcal{M}_1) \oplus H^*_{G_2}(X/G_1; \mathcal{M}_2).$$

*Proof.* Inspection of Definition 2.8 and the definition of $C^*_G(X; \mathcal{M})$. □

Proposition 2.9 easily extends to arbitrary sums of coefficient functors. The coefficient functors $\mathcal{H}^1_G(A)$ defined in 2.1 behave nicely with respect to sums.

**Proposition 2.10.** *Let $X_1, \dots, X_n$ be $G_1, \dots, G_n$-spaces and set $G = G_1 \times \dots G_n$. If $X \subseteq X_1 \times \dots \times X_n$ is a $G$-subspace then*

$$H^*_G(X; \mathcal{H}^1_G(A)) \cong H^*_G(X; \mathcal{H}^1_{G_1}(A) \oplus \dots \oplus \mathcal{H}^1_{G_n}(A)).$$

*Proof.* Denote $\mathcal{H}_G = \mathcal{H}^1_G(A)$ and $\mathcal{H}_{G_i} = \mathcal{H}^1_{G_i}(A)$. The orbits of $X$ have the form $U = (G_1/K_1) \times \dots \times (G_n/K_n) = G/(K_1 \times \dots \times K_n)$. Therefore, with the notation of Definitions 2.1 and 2.8

$$\mathcal{H}_G(U) = \mathrm{Hom}(\prod_{i=1}^n K_i, A) = \prod_{i=1}^n \mathrm{Hom}(K_i, A) =$$
$$= \oplus_{i=1}^n \mathcal{H}_{G_i}(U_{(i)}) = (\oplus_{i=1}^n \mathcal{H}_{G_i})(U).$$

It follows that $C^*_G(X; \mathcal{H}_G) \cong C^*_G(X; \mathcal{H}_{G_1} \oplus \dots \oplus \mathcal{H}_{G_n})$. □

Consider $N \lhd G$ and set $K = G/N$. Fix $G$-modules $M$ and $U$ and let $M \to I^\bullet$ be an injective resolution. Since $\mathbb{Z}[G]$ is a free right $N$-module, $I^\bullet \downarrow_N^G = \mathrm{Hom}_G(\mathbb{Z}[G], I^\bullet)$ is an injective resolution for $M \downarrow_N^G$. Since $\mathrm{Ext}^*_G(U, M)$ and $\mathrm{Ext}^*_N(U, M)$ are the homology groups of the cochain complexes $\mathrm{Hom}_G(U, I^\bullet) \subseteq \mathrm{Hom}_N(U, I^\bullet)$, the groups $\mathrm{Ext}^*_N(U, M)$ are $K$-modules and there are natural maps

$$\mathrm{Ext}^*_G(U, M) \to \mathrm{Ext}^*_N(U, M) \xrightarrow{\subseteq} \mathrm{Ext}^*_N(U, M)^K.$$

**Definition 2.11.** Suppose that $N \lhd G$ and set $K = G/N$. For a $G$-module $M$ define the following coefficient functors for $G$, where $X$ is a $G$-set

$$\begin{aligned} \mathcal{H}^i_{G|N}(M) : X &\mapsto \mathrm{Ext}^i_N(\mathbb{Z}[X], M) \\ \mathrm{fix}_K(\mathcal{H}^i_{G|N}(M)) : X &\mapsto \mathrm{Ext}^i_N(\mathbb{Z}[X], M)^K. \end{aligned}$$

There are natural maps $\mathcal{H}^i_G(M) \to \mathcal{H}^i_{G|N}(M) \to \mathrm{fix}_K(\mathcal{H}^i_{G|N}(M))$.

**Proposition 2.12.** *Set $G = N \wr \Sigma_r$ and let $Y$ be an $N$-space. Let $X$ be a $G$-subspace of $Y^r$ where $G$ acts by permuting the factors and via the action of $N^r$ on them. Let $A$ be an abelian group and let $\mathcal{H}^1_G := \mathcal{H}^1_G(A)$, $\mathcal{H}^1_{\Sigma_r} := \mathcal{H}^1_{\Sigma_r}(A)$ and $\mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{G|N^r}) := \mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{G|N^r}(A))$ be the coefficient functors defined in 2.1 and 2.11. Then there is a short exact sequence of cochain complexes*

$$0 \to C^*_{\Sigma_r}(X/N^r; \mathcal{H}^1_{\Sigma_r}) \to C^*_G(X; \mathcal{H}^1_G) \to C^*_G(X; \mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{G|N^r})) \to 0.$$

The proof is deferred to Appendix B. The basic idea of the proof is, however, very simple. We have to show that by applying these coefficient functors to an orbit $[\mathbf{x}]$ of a simplex in $X$ we obtain a short exact sequence of groups. The isotropy group of $[\mathbf{x}]$ has the form $\prod_i N \wr \Sigma_{k_i}$ where $\sum_i k_i = r$. It is an elementary exercise with commutators to check that $(N \wr \Sigma_k)_{\mathrm{ab}}$ fits into an exact sequence $N^k_{\mathrm{ab}} \to N_{\mathrm{ab}} \times (\Sigma_k)_{\mathrm{ab}} \to (\Sigma_k)_{\mathrm{ab}} \to 0$ and that by taking $\Sigma_k$-invariants of the group on the left the sequence becomes short exact and split. It therefore remains exact after applying $\mathrm{Hom}(-, A)$ and it is easy to identify the terms of this short exact sequence with the groups obtained by applying $\mathcal{H}^1_G(A)$ and $\mathcal{H}^1_{\Sigma_k}(A)$ and $\mathrm{fix}_{\Sigma_k}(\mathcal{H}^1_{G|N^r}(A))$.

## 3. Reduction of the gluing problem to $\mathcal{F}$-radical collections

For any poset $\mathcal{P}$ let $S(\mathcal{P})$ be the subdivision poset. Its objects $\mathbf{x}$ are the sequences (simplices) $x_0 \lneq x_1 \lneq \cdots \lneq x_n$ in $\mathcal{P}$ ordered by inclusion, i.e. $\mathbf{x}' \leq \mathbf{x}$ if $\mathbf{x}'$ is a subsequence (a face) of $\mathbf{x}$.

**Proposition 3.1.** *Define $\pi \colon S(\mathcal{P}) \to \mathcal{P}$ by $(x_0 \lneq \cdots \lneq x_n) \mapsto x_n$. It induces, for any functor $F \colon \mathcal{P} \to \mathbf{Ab}$, a natural isomorphism $H^*(S(\mathcal{P}); \pi^* F) \cong H^*(\mathcal{P}; F)$.*

*Proof.* By inspection, for any $y \in \mathcal{P}$ the comma category $(\pi \downarrow y)$, cf. [22, §II.6], is isomorphic to $S(\mathcal{P}_{\leq y})$ whose nerve is contractible because $\mathbf{x} \leq \mathbf{x} \cup \{y\} \geq y$

gives rise to a zigzag of natural transformations from the identity to the constant functor. The result follows e.g. from [3, Ch. XI, 9.2 and 7.2] or [26, Proposition 2.3.4] or [12, 5.4]. $\square$

Let $\mathcal{L}$ be a centric linking system associated to a fusion system $\mathcal{F}$ on $S$, see [7, §1]. Let $\mathcal{C}$ be any collection of $\mathcal{F}$-centric subgroups of $S$ closed under conjugation in $\mathcal{F}$. Let $\mathcal{F}^{\mathcal{C}}$ and $\mathcal{L}^{\mathcal{C}}$ denote the full subcategories on the object set $\mathcal{C}$. Clearly $\mathcal{F}^{\mathcal{C}}$ is a poset under inclusion. By [7, Prop. 1.11] it is possible to choose lifts $\iota_P^Q$ in $\mathcal{L}$ for the inclusions $P \leq Q$ in $\mathcal{C}$ in such a way that $\iota_Q^R \circ \iota_P^Q = \iota_P^R$. This renders $\mathcal{L}^{\mathcal{C}}$ a poset as well.

The posets $S(\mathcal{F}^{\mathcal{C}})$ and $S(\mathcal{L}^{\mathcal{C}})$, whose objects $P_0 < \cdots < P_s$ we denote by $P_\bullet$, become categories where morphisms $P_\bullet \to Q_\bullet$ are isomorphisms $P_\bullet \to P'_\bullet$ in either $\mathcal{F}^{\mathcal{C}}$ or $\mathcal{L}^{\mathcal{C}}$ where $P'_\bullet$ is a sub-chain of $Q_\bullet$. See [19] for more details. The posets of isomorphism classes are denoted $[S(\mathcal{F}^{\mathcal{C}})]$ and $[S(\mathcal{L}^{\mathcal{C}})]$. Note that $[S(\mathcal{F}^{\mathcal{C}})] = [S(\mathcal{L}^{\mathcal{C}})]$ and that their nerves are contractible by [20].

The automorphism groups of $P_\bullet$ in $S(\mathcal{F}^{\mathcal{C}})$ and $S(\mathcal{L}^{\mathcal{C}})$ are denoted, as in the introduction, $\mathrm{Aut}_{\mathcal{F}}(P_\bullet)$ and $\mathrm{Aut}_{\mathcal{L}}(P_\bullet)$. By [16, Def. 1.4, Props 1.5, 2.11], $\mathrm{Aut}_{\mathcal{L}}(P_\bullet)$ is the preimage of $\mathrm{Aut}_{\mathcal{F}}(P_\bullet) \leq \mathrm{Aut}_{\mathcal{F}}(P_s)$ in $\mathrm{Aut}_{\mathcal{L}}(P_s)$. Also, if $Q_\bullet$ is a $t$-subsimplex of $P_\bullet$ then we get a monomorphism $\mathrm{Aut}_{\mathcal{L}}(P_\bullet) \to \mathrm{Aut}_{\mathcal{L}}(Q_\bullet)$ obtained by sending $\varphi \in \mathrm{Aut}_{\mathcal{L}}(P_\bullet) \leq \mathrm{Aut}_{\mathcal{L}}(P_s)$ to its restriction $\varphi|_{Q_t} \in \mathrm{Aut}_{\mathcal{L}}(Q_t)$.

Let $\Phi\colon \mathbf{C} \to \mathbf{D}$ be a functor of small categories. The over category $(d \downarrow \Phi)$ consists of the pairs $(c, \varphi)$ where $c \in \mathbf{C}$ and $\varphi\colon d \to \Phi(c)$ is a morphism in $\mathbf{D}$. A morphism $(c, \varphi) \to (c', \varphi')$ is a morphism $\psi\colon c \to c'$ such that $\psi \circ \varphi = \varphi'$. See [22, §II.6]. There is an obvious projection functor $\pi_d\colon (d \downarrow \Phi) \to \mathbf{C}$. For any functor $\mathcal{A}\colon \mathbf{C} \to \mathbf{Ab}$ there is a change of base spectral sequence, see [11, Appendix II.3] for the dual result or [19, Section 2],

$$E_2^{ij} = H^i(\mathbf{D}; d \mapsto H^j((d \downarrow \Phi); \pi_d^*(\mathcal{A}))) \Rightarrow H^{i+j}(\mathbf{C}; \mathcal{A}).$$

The edge maps $H^j(\mathbf{C}; \mathcal{A}) \to E_2^{0,j}$ are obtained by taking the inverse limit over $d \in \mathbf{D}$ of the natural maps $\pi_d^*\colon H^j(\mathbf{C}; \mathcal{A}) \to H^j((d \downarrow \Phi); \pi_d^*(\mathcal{A}))$.

Given a collection $\mathcal{C}$ of $\mathcal{F}$-centric subgroups of $S$, Linckelmann proved in [19, Theorem 1.1] the existence of the following short exact sequence which is obtained in [19, Section 3] by means of the the base change spectral sequence

$$0 \to H^1([S(\mathcal{L}^{\mathcal{C}})]; \mathcal{A}^1_{\mathcal{L}}) \to H^2(\mathcal{L}^{\mathcal{C}}; k^\times) \xrightarrow{\epsilon} H^0([S(\mathcal{L}^{\mathcal{C}})]; \mathcal{A}^2_{\mathcal{L}}) \to H^2([S(\mathcal{L}^{\mathcal{C}})]; \mathcal{A}^1_{\mathcal{L}}).$$

The map $\epsilon$ is the composition of the isomorphism $H^*(\mathcal{L}^{\mathcal{C}}; k^\times) \cong H^*(S(\mathcal{L}^{\mathcal{C}}); k^\times)$ given in Proposition 3.1 above, and the edge homomorphism $H^2(S(\mathcal{L}^{\mathcal{C}}); k^\times) \to E_2^{0,2}$ in the base-change spectral sequence of the projection $\pi\colon S(\mathcal{L}^{\mathcal{C}}) \to [S(\mathcal{L}^{\mathcal{C}})]$. By inspection $([P_\bullet] \downarrow \pi)$ is identified with a full subcategory of $S(\mathcal{L}^{\mathcal{C}})$ which contains the object $P_\bullet$ and $\mathrm{Aut}_{S(\mathcal{L}^{\mathcal{C}})}(P_\bullet)$ as an initial full subcategory. Therefore the homomorphism $\epsilon$ in Linckelmann's exact sequence is the same homomorphism defined in the introduction. Namely it is the inverse limit over $[P_\bullet] \in [S(\mathcal{L}^{\mathcal{C}})]$ of the restriction homomorphisms $H^2(\mathcal{L}^{\mathcal{C}}; k^\times) \to H^2(\mathrm{Aut}_{S(\mathcal{L}^{\mathcal{C}})}(P_\bullet); k^\times)$ where $\mathrm{Aut}_{S(\mathcal{L}^{\mathcal{C}})}(P_\bullet)$ is identified with a subcategory of $\mathrm{Aut}_{\mathcal{L}}(P_s) \subseteq \mathcal{L}^{\mathcal{C}}$.

***Proof of Theorem 1.2.*** Let $[P_\bullet]$ denote the $\mathcal{F}$-conjugacy class of an $s$-simplex $P_0 < \cdots < P_s$ in $\mathcal{F}^{\mathcal{C}}$. The kernel of the epimorphism $\mathrm{Aut}_{\mathcal{L}}(P_\bullet) \to \mathrm{Aut}_{\mathcal{F}}(P_\bullet)$ is $Z(P_s)$ and since $k^\times$ has no $p$-torsion, $\tilde{H}^*(Z(P_s); k^\times) = 0$. The Leray-Serre spectral sequence now implies that $\mathcal{A}^i_{\mathcal{F}}([P_\bullet]) = \mathcal{A}^i_{\mathcal{L}}([P_\bullet])$ for all $i$. In particular

$$\varprojlim_{[S(\mathcal{F}^{\mathcal{C}})]}{}^{*} \mathcal{A}^i_{\mathcal{L}} \cong \varprojlim_{[S(\mathcal{F}^{\mathcal{C}})]}{}^{*} \mathcal{A}^i_{\mathcal{F}}.$$

Since $k^\times$ is a divisible group with no $p$-torsion it follows from [6, Lemma 1.3] by applying it to every prime $q \neq p$ at a time[1], that $H^*(\mathcal{L}^{\mathcal{C}}; k^\times) \cong H^*(\mathcal{F}^{\mathcal{C}}; k^\times)$. Here we used $H^*(\mathbf{C}; k^\times) \cong H^*(\mathbf{C}^{\mathrm{op}}; k^\times)$ for every category $\mathbf{C}$. This proves the equivalence of (c) and (d). The equivalence of (a) and (b) is a special case when $\mathcal{C}$ is the collection of all the $\mathcal{F}$-centric subgroups.

Given the collection $\mathcal{C}$ we obtain the following diagram in which the square commutes and the bottom row is exact by [19, Theorem 1.1].

$$\begin{tikzcd}
H^2(\mathcal{L}; k^\times) \arrow[r, "\epsilon"] \arrow[d, "\mathrm{res}"'] & \varprojlim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{L}} \arrow[d, "\mathrm{res}"] & \\
H^2(\mathcal{L}^{\mathcal{C}}; k^\times) \arrow[r, "\epsilon"] & \varprojlim_{[S(\mathcal{F}^{\mathcal{C}})]} \mathcal{A}^2_{\mathcal{L}} \arrow[r] & H^2([S(\mathcal{F}^{\mathcal{C}})]; \mathcal{A}^1_{\mathcal{L}})
\end{tikzcd}$$

We will prove below that the first vertical arrow is an isomorphism and that the second is a monomorphism. The equivalence of (b) and (c) then follows by a simple diagram chasing. If the third group in the second row vanishes, then the first arrow is an epimorphism which implies that the gluing problem has a solution in $\mathcal{L}^{\mathcal{C}}$.

Since $\mathcal{C}$ contains the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of $S$, the inclusion $|\mathcal{L}^{\mathcal{C}}| \subseteq |\mathcal{L}|$ is a homotopy equivalence by [5, Theorem 3.5]. It follows that $H^2(\mathcal{L}; k^\times) \xrightarrow{\mathrm{res}} H^2(\mathcal{L}^{\mathcal{C}}; k^\times)$ is an isomorphism because $H^*(\mathcal{L}^{\mathcal{C}}; k^\times) \cong H^*(|\mathcal{L}^{\mathcal{C}}|; k^\times)$.

Consider an element $(\kappa_{[P_\bullet]})_{[P_\bullet] \in [S(\mathcal{F}^c)]}$ in the kernel of

$$\varprojlim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{L}} \xrightarrow{\mathrm{res}} \varprojlim_{[S(\mathcal{F}^{\mathcal{C}})]} \mathcal{A}^2_{\mathcal{L}}.$$

Thus, $(\kappa_{[P_\bullet]})_{[P_\bullet] \in [S(\mathcal{F}^c)]}$ is a compatible family of classes in $\mathcal{A}^2_{\mathcal{L}}([P_\bullet])$ such that $\kappa_{[P_\bullet]} = 0$ if $P_0, \ldots, P_s \in \mathcal{C}$. Our goal is to prove that $\kappa_{[P_\bullet]} = 0$ for all $P_\bullet$. The morphism $[P_0] \to [P_\bullet]$ in $[S(\mathcal{F}^c)]$ carries $\kappa_{[P_0]}$ to $\kappa_{[P_\bullet]}$ since the family is compatible so it suffices to prove that $\kappa_{[P]} = 0$ for all $P \in \mathcal{L}$.

Let $\mathcal{E}$ denote the collection of all $P \in \mathcal{L}$ such that $\kappa_{[P]} \neq 0$. Clearly $\mathcal{E}$ must be disjoint from $\mathcal{C}$. We want to prove that $\mathcal{E}$ is empty. If this is not the case, choose $P \in \mathcal{E}$ which has maximal order. We may assume that $P$ is fully normalised. Note that $P \notin \mathcal{C}$ so $P$ is not $\mathcal{F}$-radical, hence $Q = O_p(\mathrm{Aut}_{\mathcal{L}}(P)) \gneq P$. Since $P$

[1] The proof of [6, Lemma 1.3] applies verbatim for $q = 0$ which is needed if $\mathbb{Q} \leq k^\times$, i.e. $k \supsetneq \bar{\mathbb{F}}_p$

is maximal, $Q \notin \mathcal{E}$ whence $\kappa_{[Q]} = 0$. We have inclusion of groups, see [16, Prop. 1.5],

$$\mathrm{Aut}_{\mathcal{L}}(P) \;\hookleftarrow\; \mathrm{Aut}_{\mathcal{L}}(P < Q) \;\leq\; \mathrm{Aut}_{\mathcal{L}}(Q).$$

Since $Q$ is the preimage of $O_p(\mathrm{Aut}_{\mathcal{F}}(P))$ in $S$, the map $\mathrm{Aut}_{\mathcal{F}}(P < Q) \to \mathrm{Aut}_{\mathcal{F}}(P)$ is surjective by [13, Lemma 2.4]. It follows that $\mathrm{Aut}_{\mathcal{L}}(P < Q) \to \mathrm{Aut}_{\mathcal{L}}(P)$ is surjective by [7, Lemma 1.10(a)], and therefore it is an isomorphism. Thus, the morphisms $[P] \xrightarrow{\varphi} [P < Q] \xleftarrow{\psi} [Q]$ in $[S(\mathcal{F}^c)]$ induce

$$\mathcal{A}^i_{\mathcal{L}}([P]) \xrightarrow[\cong]{\varphi_*} \mathcal{A}^i_{\mathcal{L}}([P < Q]) \xleftarrow{\psi_*} \mathcal{A}^i_{\mathcal{L}}([Q]).$$

Compatibility implies that $\varphi^*(\kappa_{[P]}) = \kappa_{[P<Q]} = \psi_*(\kappa_{[Q]}) = 0$, whence $\kappa_{[P]} = 0$. This is a contradiction to the choice of $P$. $\square$

Recall that a $p$-subgroup $P$ of a finite group $G$ is called $p$-centric if $Z(P)$ is a Sylow $p$-subgroup of $C_G(P)$. Alternatively, $C_G(P) = Z(P) \times C'_G(P)$ where the order of $C'_G(P)$ is not divisible by $p$. We say that $P$ is $p$-radical if $O_p(N_G(P)) = P$, namely if $P$ is the maximal normal $p$-subgroup of $N_G(P)$.

**Definition 3.2.** Let $S_p(G)$ denote the poset of the $p$-subgroups of $G$ with action by conjugation. A *collection* $\mathcal{D}$ of $p$-subgroups is a $G$-subposet of $S_p(G)$. Let $S_p^c(G)$ denote the collection of the $p$-centric subgroups and let $S_p^r(G)$ denote the collection of the $p$-radical subgroups. Set $S_p^{rc}(G) \overset{\text{def}}{=} S_p^r(G) \cap S_p^c(G)$.

**Proposition 3.3** ([8, Example 3.3])**.** *Let $S$ be a Sylow $p$-subgroup of a finite group $G$ and let $\mathcal{F} = \mathcal{F}_S(G)$ be the associated fusion system. A subgroup $P \leq S$ is $\mathcal{F}$-centric if and only if it is $p$-centric. If $P$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical then it is $p$-centric and $p$-radical in $G$.*

**Corollary 3.4.** *Let $\mathcal{F} = \mathcal{F}_S(G)$ be the fusion system of a finite group $G$ with a Sylow subgroup $S$ and set $\mathcal{H} := \mathcal{H}^1_G(k^\times)$, see Definition 2.1. Let $\mathcal{D}$ be a collection of $p$-subgroups of $G$ such that $S_p^{rc}G) \subseteq \mathcal{D} \subseteq S_p^c(G)$ and suppose that every $P \in \mathcal{D}$ is centric, namely $C_G(P) = Z(P)$.*

*If $H^2_G(|\mathcal{D}|; \mathcal{H}) = 0$ then the gluing problem in $\mathcal{F}^c$ has a solution for any family of compatible classes $\alpha_{[P_\bullet]} \in \mathcal{A}^2_{\mathcal{F}}([P_\bullet])$.*

*Proof.* Let $\mathcal{C}$ denote the poset $\{P \in \mathcal{D} \colon P \leq S\}$. By Proposition 3.3, $\mathcal{C}$ consists of $\mathcal{F}$-centric subgroups and contains all the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of $S$. It is clearly closed under $\mathcal{F}$-conjugation. By [17, Prop. 3.2], the groups $H^*([S(\mathcal{F}^{\mathcal{C}})]; \mathcal{A}^1_{\mathcal{L}})$ are isomorphic to the homology groups of the cochain complex $E^*$ where

$$E^n = \prod_{[P_0 < \cdots < P_n] \in [S(\mathcal{F}^{\mathcal{C}})]} \mathcal{A}^1_{\mathcal{L}}([P_0 < \cdots < P_n]),$$

and coface maps are the usual alternating sums of the maps obtained from the inclusion of faces of $n$-simplices. Note that $[S(\mathcal{F}^{\mathcal{C}})]$ is the poset of the non-degenerate simplices of $|\mathcal{D}|/G$ because $\mathcal{C}$ contains an element from every $G$-conjugacy class of $\mathcal{D}$ and since conjugation of elements of $\mathcal{C}$ in $G$ and in $\mathcal{F}$ are the same thing. Also, by hypothesis $\mathrm{Aut}_{\mathcal{L}}(P) = N_G(P)$ for any $P \in \mathcal{D}$. Hence, $E^* \cong NC^*_G(|\mathcal{D}|; \mathcal{H})$ because

$$\mathcal{H}([P_\bullet]) = \mathrm{Hom}(\cap_i N_G(P_i), k^\times) = \mathrm{Hom}(\mathrm{Aut}_{\mathcal{L}}(P_0 < \cdots < P_k), k^\times) = \mathcal{A}^1_{\mathcal{L}}([P_\bullet]).$$

We deduce that $H^2([S(\mathcal{F}^{\mathcal{C}})]; \mathcal{A}^1_{\mathcal{L}}) = H^2(E^*) \cong H^2_G(|\mathcal{D}|; \mathcal{H}) = 0$ by hypothesis on $|\mathcal{D}|$. The result follows from Theorem 1.2. □

## 4. Basic $p$-subgroups of the symmetric group and partitions

In this section we make several definitions and observations which will be the fundamental building blocks in proving the hypotheses of Corollary 3.4 for the fusion systems of the symmetric and alternating groups. Most of the results, if not all of them, are probably known but we were unable to find published references and we therefore prove them in Appendix E.

**Definition 4.1** (Alperin-Fong, [1])**.** Fix a prime $p$. Let $(\mathbb{Z}/p)^c$ act on itself by left translations and let $V_c$ denote its image in $\Sigma_{p^c}$ via this action.

A $p$-subgroup $P \leq \Sigma_n$ is called *basic* if it is conjugate to

$$V_{c_1,\dots,c_t} \overset{\text{def}}{=} V_{c_1} \wr V_{c_2} \wr \cdots \wr V_{c_t} \qquad (\leq \Sigma_{p^{c_1+\cdots+c_t}}.)$$

Its *degree* is $\deg(P) = c_1 + \cdots + c_t$.

**Definition 4.2.** Let $\mathcal{B}(d)$ be the poset of the basic subgroups of degree $d$ in $\Sigma_{p^d}$. For $r \geq 1$ let $\mathcal{B}(d)^r$ be the $r$-fold product and let $\mathcal{B}(d)^r_0$ denote the $\Sigma_{p^d} \wr \Sigma_r$-poset $\mathcal{B}(d)^r$ with the conjugacy class of $V_d \times \cdots \times V_d$ removed.

**Definition 4.3.** An *interval* in $\underline{d} = \{1, \dots, d\}$ is a *non-empty* subset of the form $\{x : a \leq x \leq b\}$ for some $a, b$. Let $\mathcal{P}art(d)$ denote the poset of all the partitions $\mathbf{c}$ of $\underline{d}$ into intervals where $\mathbf{c} \leq \mathbf{c}'$ if $\mathbf{c}'$ is a refinement of $\mathbf{c}$. We write partitions in the form $\mathbf{c} = (c_1, \dots, c_t)$ where $c_i$ is the length of the $i^{\text{th}}$ interval and $c_1 + \cdots + c_t = d$.

Define the following partitions consisting of $d$, 1 intervals,

$$\lambda^{\max} = (1, 1, \dots, 1), \qquad \lambda^{\min} = (d).$$

For $1 \leq i \leq d-1$ define the following partitions consisting of 2 and $d-1$ intervals

$$\lambda_i = (i, d-i) \quad \text{and} \quad \widehat{\lambda_i} = (\underbrace{1, \dots, 1}_{i-1 \text{ terms}}, 2, \underbrace{1, \dots, 1}_{d-i-1 \text{ terms}})$$

Given $r \geq 1$ let $\mathcal{P}art(d)_r$ denote the $\Sigma_r$-poset $\mathcal{P}art(d)^r - \{(\lambda^{\min}, \dots, \lambda^{\min})\}$.

**Definition 4.4.** Let $\mathrm{GL}_{\mathbf{c}}(p)$, where $\mathbf{c} = (c_1, \dots, c_t) \in \mathcal{P}art(d)$ denote the group $\prod_{i=1}^t \mathrm{GL}_{c_i}(p)$ as a subgroup of $\mathrm{GL}_d(p)$.

It is clear that the assignment $\mathbf{c} \mapsto \mathrm{GL}_{\mathbf{c}}(p)$ is order reversing, namely if $\mathbf{c} \leq \mathbf{c}'$ then $\mathrm{GL}_{\mathbf{c}'}(p) \leq \mathrm{GL}_{\mathbf{c}}(p)$. Let $\mathrm{subs}(G)$ denote the poset of the subgroups of a group $G$. The main results of this section are the next proposition and lemma.

**Proposition 4.5.** *Set $n = p^d$. There are functions $V\colon \mathcal{P}art(d) \to \mathcal{B}(d)$ and $R\colon \mathcal{P}art(d) \to \mathrm{subs}(\Sigma_n)$ with the following properties.*

*(i) $V$ is order preserving and if $\mathbf{c} = (c_1, \dots, c_t)$ then $V(\mathbf{c})$ is a basic subgroup conjugate to $V_{c_1,\dots,c_t}$. Thus, $\mathcal{P}art(d)$ can be identified as a subposet of $\mathcal{B}(d)$ via the injection $V$.*

*(ii) The composite $|\mathcal{P}art(d)| \to |\mathcal{B}(d)| \to |\mathcal{B}(d)|/\Sigma_n$ is an isomorphism of simplicial sets.*

*(iii) $R$ is order reversing and there are isomorphisms $R(\mathbf{c}) \cong \mathrm{GL}_{\mathbf{c}}(p)$ such that if $\mathbf{c} \le \mathbf{c}'$, then $R(\mathbf{c}') \le R(\mathbf{c})$ is the inclusion $\mathrm{GL}_{\mathbf{c}'}(p) \le \mathrm{GL}_{\mathbf{c}}(p)$. Moreover*

$$N_{\Sigma_n}(V(\mathbf{c})) = V(\mathbf{c}) \rtimes R(\mathbf{c})$$

*and if $p > 2$ then every factor $\mathrm{GL}_{c_i}(p) \le R(\mathbf{c})$ contains odd permutations.*

**Lemma 4.6.** *Every $P \in \mathcal{B}(d)$ is centric, namely $C_{\Sigma_{p^d}}(P) = Z(P)$.*

*Proof.* This follows from [1, (1.2)] since $C_{\Sigma_{p^d}}(V_d) = V_d$. □

Recall that the support of an element $g \in \Sigma_n$, denoted $\mathrm{supp}(g)$, is the set of points which $g$ does not fix. For a subgroup $G \le \Sigma_n$ set $\mathrm{supp}(G) = \cup_{g \in G} \mathrm{supp}(g)$, that is, $\mathrm{supp}(G)$ is the complement of the set $\mathrm{fix}(G)$ of the fixed points of $G$. The following definition and lemmas should be compared with [21, Lemma 5]. They will be fundamental tools in Sections 5, 6 and 7.

**Definition 4.7.** Fix any integer $r \ge 0$ and any $K \le \Sigma_n$, let $\Delta_r(K)$ be the set of $g \in K$ such that $|\mathrm{supp}(g)| \le r$. Let $\delta_r(K)$ be the subgroup generated by $\Delta_r(K)$.

**Lemma 4.8.** *For any $K \le \Sigma_n$ we have $\delta_r(K) \le K$ and moreover $\delta_r(K) \lhd N_{\Sigma_n}(K)$. In addition $\delta_r(K_1 \times \cdots \times K_s) = \delta_r(K_1) \times \cdots \times \delta_r(K_s)$ and $\delta_r(H) \le \delta_r(K)$ if $H \le K$.*

*Proof.* Immediate consequence of the definitions. □

**Lemma 4.9.** *Consider a basic subgroup $P = V_{c_1,\dots,c_t}$ of degree $d$ and some $r \ge 1$. Let $q$ be the largest integer $\le t$ such that $s := p^{c_1+\cdots+c_q} \le r$ and set $e = p^{c_{q+1}+\cdots+c_t}$. Then $\delta_r(P) = (V_{c_1,\dots,c_q})^e \lhd P$. Moreover, $\delta_r(P)$ contains every basic subgroup of degree $\le \log_p r$ which is contained in $P$.*

*Proof.* If $q = t$ then $\delta_r(P) = P$ and the result is trivial. Assume that $q < t$ and set $H = V_{c_1,\dots,c_q}$ and $K = V_{c_{q+1},\dots,c_t}$. Then $P = H \wr K$ and $K = V_{c_{q+1}} \wr K'$ and $H$ is transitive on $s$ points. Now $V_{c_{q+1}}$ acts transitively and freely on $p^{c_{q+1}}$ elements so by Lemma A.1, $|\mathrm{supp}(k)| \ge p^{c_{q+1}}$ for any $1 \ne k \in K$. By the same lemma, if $g \in P - H^e$ then $|\mathrm{supp}(g)| \ge sp^{c_{q+1}} = p^{c_1+\cdots+c_{q+1}} > r$, hence $g \notin \Delta_r(P)$. It follows that $\delta_r(P) \subseteq H^e$ and the reverse inclusion is clear from Lemma 4.8. Finally, if $U \le P$ is basic of degree $\le \log_p r$ then $U = \delta_r(U) \le \delta_r(P)$. □

## 5. Proof of Theorem 1.1 for the fusion system of $\Sigma_n$.

Alperin and Fong proved in [1] that if $P$ is a $p$-radical $p$-subgroup of $\Sigma_n$ then it is a product of basic subgroups. The proof of Theorem 1.1 for the fusion system of $\Sigma_n$ is an inductive procedure based on Corollary 3.4 and 5.1–5.4 below.

**Definition 5.1.** Let $\mathcal{D}(n)$ be the collection of the $p$-subgroups $P \leq \Sigma_n$ which are products of basic subgroups (Definition 4.1) and which fix $p-1$ points *or less.*

**Proposition 5.2.** *There are inclusions $S_p^{rc}(\Sigma_n) \subseteq \mathcal{D}(n) \subseteq S_p^c(\Sigma_n)$ (Definition 3.2). If $p \mid n$ then every $P \in \mathcal{D}(n)$ is centric, namely $C_{\Sigma_n}(P) = Z(P)$.*

**Definition 5.3.** The *components* of $P \in \mathcal{D}(n)$ are the basic subgroups whose product is $P$. Let $\deg(P)$ be the maximum degree of the components of $P$.

(a) Let $\mathcal{D}_d(n)$ be the subposet of $\mathcal{D}(n)$ containing $P$ such that $\deg(P) \leq d$.

(b) For $r \geq 0$ let $\mathcal{E}_{d,r}(n)$ be the subposet of $\mathcal{D}_d(n)$ of those subgroups $P$ such that *at most* $r$ of its components are conjugate to $V_d$, see Definition 4.1.

It is clear that $\mathcal{D}_{d-1}(n) \subseteq \mathcal{E}_{d,0}(n) \subseteq \mathcal{E}_{d,1}(n) \subseteq \dots$ and that if $r$ is large enough, namely if $(r+1)p^d > n$, then $\mathcal{E}_{d,r}(n) = \mathcal{D}_d(n)$ because every $V_d$ supports $p^d$ elements. If $d$ is large enough, specifically if $p^d \geq n$, then $\mathcal{D}_d(n) = \mathcal{D}(n)$ because $\Sigma_n$ cannot contain any basic subgroup of degree $> d$.

**Proposition 5.4.** *Fix $n \geq p$, $d \geq 1$ and $r \geq 0$.*

*(a) $|\mathcal{D}_1(n)|$ is a transitive (discrete) $\Sigma_n$-set.*

*(b) For any $d \geq 2$ the inclusion $|\mathcal{D}_{d-1}(n)| \subseteq |\mathcal{E}_{d,0}(n)|$ is a $\Sigma_n$-homotopy equivalence.*

*(c) If $d \geq 2$ and $r \geq 1$ then*

$$H^{*\geq 2}_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathcal{H}^1_{\Sigma_n}(k^\times)\,) = 0$$

*(d) $|\mathcal{E}_{d,r}(n)|/\Sigma_n$ is acyclic if either $d \geq 2$ or if $d = 1$ and $(r+1)p > n$.*

***Proof of Theorem 1.1 for*** $\Sigma_n$. The fusion system of $\Sigma_n$ is isomorphic to the fusion system of $\Sigma_m$ if $p \mid m$ and $0 \leq n - m < p$. In light of Corollary 3.4 and Proposition 5.2, it remains to prove that if $p \mid n$ and $k$ is an algebraically closed field of characteristic $p$, then

$$H^*_{\Sigma_n}(|\mathcal{D}(n)|; \mathcal{H}^1_{\Sigma_n}(k^\times)\,) = 0 \qquad \text{for all } * \geq 2.$$

Now, $\mathcal{D}(n) = \mathcal{D}_d(n)$ for large $d$ so it remains to prove by induction that

$$(\dagger) \qquad H^{*\geq 2}_{\Sigma_n}(|\mathcal{D}_d(n)|; \mathcal{H}^1_{\Sigma_n}(k^\times)\,) = 0$$

for all $d \geq 1$. The base of induction $d = 1$ is immediate from Proposition 5.4(a). Assume that ($\dagger$) holds for $d \geq 1$. Induction on $r$, using Proposition 5.4(b) and (c), shows that $H^{*\geq 2}(|\mathcal{E}_{d+1,r}(n)|; \mathcal{H}^1_{\Sigma_n}(k^\times)) = 0$ for all $r \geq 0$. The induction step for $d+1$ follows since $\mathcal{D}_{d+1}(n) = \mathcal{E}_{d+1,r}(n)$ for sufficiently large $r$. $\square$

We will now prove Propositions 5.2 and 5.4.

**Proposition 5.5.** *Consider a subgroup $P = P_1^{r_1} \times \cdots \times P_k^{r_k}$ of $\Sigma_n$ where $P_1, \dots, P_k$ are non-conjugate basic $p$-subgroups of degrees $d_1, \dots, d_k$. Set $m = |\operatorname{fix}(P)|$. Then*

$$C_{\Sigma_n}(P) = \prod_{i=1}^{k} Z(P_i)^{r_i} \times \Sigma_m \qquad \textit{and} \qquad N_{\Sigma_n}(P) = (\prod_{i=1}^{k} N_{\Sigma_{p^{d_i}}}(P_i) \wr \Sigma_{r_i}) \times \Sigma_m.$$

*Proof.* Lemma 4.6 implies the first equality, the second is in [1, Section 2B]. □

***Proof of Proposition 5.2.*** Fix $P \in \mathcal{D}(n)$. By Proposition 5.5, $C(P) = Z(P) \times \Sigma_m$ where $m \le p-1$, hence $P$ is $p$-centric. If $p \mid n$ then $m = 0$ because every basic subgroup $P_i$ is transitive on $p^{d_i}$ elements, and therefore $P$ contains its centraliser.

If $P$ is a $p$-centric and $p$-radical subgroup of $\Sigma_n$ the by [1, Section (2A)] it is a product of basic $p$-subgroups and since it is $p$-centric, Proposition 5.5 implies that $|\operatorname{fix}(P)| < p$, namely $P \in \mathcal{D}(n)$. □

**Definition 5.6.** Fix $d \ge 2$, $r \ge 1$ and set $n = rp^d$. Define the following $\Sigma_n$-posets

(a) $\mathcal{B}(d,r) = \{P \in \mathcal{D}(n) \colon P$ is a product of $r$ basic subgroups of degree $d\}$.

(b) $\mathcal{V}(d,r)$ is the conjugacy class of $V_d \times \cdots \times V_d$ ($r$ factors).

(c) $\mathcal{B}_0(d,r) = \mathcal{B}(d,r) - \mathcal{V}(d,r)$.

Note that $\mathcal{B}(d,1) = \mathcal{B}(d)$, see Definition 4.2.

**Proposition 5.7.** *Fix $d \ge 2$ and $r \ge 1$ and set $n = rp^d$. Then*

*(i) The spaces $|\mathcal{B}(d,r)|/\Sigma_n$ and $|\mathcal{B}_0(d,r)|/\Sigma_n$ are contractible.*

*(ii) $H^*_{\Sigma_n}(|\mathcal{B}(d,r)|, |\mathcal{B}_0(d,r)|; \mathcal{H}^1_{\Sigma_n}(k^\times)) = 0$ for all $* \ge 2$ (See Definition 2.1).*

*Proof.* Set $K = \Sigma_{p^d}$ and $\Gamma = K \wr \Sigma_r$. Let $\mathcal{V}(d)$ be the conjugacy class of $V_d$ in $\mathcal{B}(d)$. Thus, $\mathcal{B}(d)_0^r = \mathcal{B}(d)^r - \mathcal{V}(d)^r$ is a $\Gamma$-poset, see Definition 4.2.

The high transitivity of $\Sigma_n$ implies that any $P \in \mathcal{B}(d,r)$ is conjugate to a subgroup of $(\Sigma_{p^d})^r$, namely $P$ is conjugate to some $Q \in \mathcal{B}(d)^r$. By looking at the orbits of the basic subgroups, it follows that if $Q, Q' \in \mathcal{B}(d)^r$ are conjugate via some $g \in \Sigma_n$, then $g \in \Gamma$. In particular $N_{\Sigma_n}(Q) \le \Gamma$. By Proposition 2.5,

$$\mathcal{B}(d,r) = \mathcal{B}(d)^r \uparrow_\Gamma^{\Sigma_n}, \qquad \text{and} \qquad \mathcal{B}_0(d,r) = \mathcal{B}(d)_0^r \uparrow_\Gamma^{\Sigma_n}.$$

Note that $\mathcal{P}art(d)^r$ and $\mathcal{P}art(d)_0^r$, see Definition 4.3, have $\Sigma_r$-contractible nerves because $d \ge 2$ so they have a maximum. We deduce from Proposition 4.5(ii) that

$$|\mathcal{B}(d,r)|/\Sigma_n \cong |\mathcal{B}(d)^r|/\Gamma = |\mathcal{P}art(d)^r|/\Sigma_r \simeq \mathrm{pt}\,.$$

Similarly $|\mathcal{B}_0(d,r)|/\Sigma_n = |\mathcal{B}(d)_0^r|/\Gamma = |\mathcal{P}art(d)_0^r|/\Sigma_r \simeq *$. This proves point (i). Proposition 2.7 together with Definition 2.6 imply that

$$H^*_{\Sigma_n}(|\mathcal{B}(d,r)|;\mathcal{H}^1_{\Sigma_n}(k^\times)) = H^*_\Gamma(|\mathcal{B}(d)^r|;\mathcal{H}^1_\Gamma(k^\times)) \qquad \text{and}$$
$$H^*_{\Sigma_n}(|\mathcal{B}_0(d,r)|;\mathcal{H}^1_{\Sigma_n}(k^\times)) = H^*_\Gamma(|\mathcal{B}(d)_0^r|;\mathcal{H}^1_\Gamma(k^\times)).$$

In light of Propositions 2.12 and 2.3 and the $\Sigma_r$-contractibility of $|\mathcal{P}art(d)^r|$ and $|\mathcal{P}art(d)_0^r|$, we see that in order to prove point (ii) it remains to prove that

$$H^{*\geq 1}_\Gamma(|\mathcal{B}(d)^r|;\mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{\Gamma|K^r}(k^\times))) = H^{*\geq 1}_\Gamma(|\mathcal{B}(d)_0^r|;\mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{\Gamma|K^r}(k^\times))) = 0.$$

This is shown in Proposition C.1 whose proof only depends on the results in sections 2 and 4. □

**Proposition 5.8.** *If $r \geq 1$ and $rp^d \leq n$ then there is a relative isomorphism*

$$(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|) \cong (|\mathcal{B}(d,r)\times\mathcal{E}_{d,0}(m)|, |\mathcal{B}_0(d,r)\times\mathcal{E}_{d,0}(m)|)\uparrow^{\Sigma_n}_{\Sigma_{n'}\times\Sigma_m}$$

*where $n' = rp^d$ and $m = n - n'$. The isotropy group of a $k$-simplex $Q_\bullet \times R_\bullet$ in $|\mathcal{B}(d,r)\times\mathcal{E}_{d,0}(m)| - |\mathcal{B}_0(d,r)\times\mathcal{E}_{d,0}(m)|$ is conjugate to the following subgroup of $\Sigma_{n'}\times\Sigma_m$*

$$(\prod_{i=1}^s N_{\Sigma_{p^{d_i}}}(Q_\bullet^{(i)}) \wr \Sigma_{e_i}) \times \mathrm{Iso}_{\Sigma_m}(R_\bullet)$$

*where $Q_\bullet^{(1)},\dots,Q_\bullet^{(s)}$ are non-conjugate $k$-simplices in $|\mathcal{B}(d_i)|$, see Definition 4.2. Also, $Q_0^{(i)} = V_d$ for all $i$ and $e_1+\dots+e_s = r$ and $R_\bullet$ should be removed from the expression if and only if $m < p$.*

*Proof.* If $rp^d \leq n$ then a $k$-simplex $P_\bullet$ of the form $P_0 \leq \dots \leq P_k$ in $|\mathcal{E}_{d,r}(n)| - |\mathcal{E}_{d,r-1}(n)|$ has the following properties. First, each $P_i$ has *at most* $r$ components which are conjugate to $V_d$. Second, there exists some $i$ such that $P_i \notin \mathcal{E}_{d,r-1}(n)$, thus, the number of component of $P_i$ that are conjugate to $V_d$ is exactly $r$. By Lemma 4.9, $V_d$ contains no proper basic subgroup so $P_0$ has exactly $r$ factors conjugate to $V_d$. By looking at the orbits of the components of the groups $P_i$ we see that for every component $V_d$ of $P_0$ there must be a component of degree $d$ in every $P_i$ that contains it because there is a bijection between the orbits of $P_i$ and its components. The other components of $P_i$ cannot be conjugate to $V_d$ (otherwise $P_0$ will have $r+1$ such components). Thus, up to conjugation in $\Sigma_n$, we may assume that $P_\bullet$ belongs to the relative $\Sigma_{n'}\times\Sigma_m$-space $(|\mathcal{B}(d,r)\times\mathcal{E}_{d,0}(m)|, |\mathcal{B}_0(d,r)\times\mathcal{E}_{d,0}(m)|)$. If $P'_\bullet$ in this relative space is $\Sigma_n$-conjugate to $P_\bullet$ then they are conjugate in $\Sigma_{n'}\times\Sigma_m$ because the components $V_d$ of $P_0$ can only be conjugate to these components in $P'_0$. The first result follows from Proposition 2.5.

We can now write $P_\bullet$ in the form $Q_\bullet \times R_\bullet$ where $Q_\bullet \in |\mathcal{B}(d,r)|$ and $R_\bullet \in |\mathcal{E}_{d,0}(m)|$. By the definition of $\mathcal{D}(n)$ it is clear that $R_\bullet$ should be suppressed if and only if $m < p$. We may assume that $Q_0 = (V_d)^r$ and since $Q_i$ must be a product of basic subgroups of degree $d$, by looking at the orbits of $Q_0$ we see that $Q_\bullet$ is conjugate in $\Sigma_{n'}$ to $(Q_\bullet^{(1)})^{e_1} \times \dots (Q_\bullet^{(s)})^{e_s}$ where $Q_\bullet^{(i)}$ are non-conjugate $k$-simplices in $|\mathcal{B}(d)|$. Proposition 5.5 now easily implies the formula for $\mathrm{Iso}_{\Sigma_n}(P_\bullet)$. $\square$

***Proof of Proposition 5.4.*** (a) Any $P \in \mathcal{D}_1(n)$ must be conjugate in $\Sigma_n$ to $(V_1)^r$ where $0 \le n - rp < p$ by [1, (2A)].
(b) If $Q = V_{c_1,\dots,c_t}$ of degree $d$ is different from $V_d$ then $t > 1$ and by Lemma 4.9, $\delta_{p^{d-1}}(Q) = (V_{c_1,\dots,c_{t-1}})^{p^{c_t}}$ is a product of basic subgroups of degree $\le d-1$ and $\mathrm{supp}(\delta_{p^{d-1}}(Q)) = \mathrm{supp}(Q)$. If $\deg(Q) \le d-1$ then $\delta_{p^{d-1}}(Q) = Q$. By Lemma 4.8 and the definition of $\mathcal{D}_{d-1}(n)$ in 5.3, we obtain a map of $\Sigma_n$-posets

$$\mathcal{E}_{d,0}(n) \xrightarrow{\delta_{p^{d-1}}} \mathcal{D}_{d-1}(n).$$

Since $\delta_{p^{d-1}}(P) \le P$, we obtain a natural transformation $i \circ \delta_{p^{d-1}} \to \mathrm{Id}$. Also, $\delta_{p^{d-1}} \circ i = \mathrm{Id}$ where $i$ is the inclusion $\mathcal{D}_{d-1}(n) \subseteq \mathcal{E}_{d,0}(n)$. Therefore $|i|$ is a $\Sigma_n$-equivariant homotopy equivalence.
(d) The case $d = 1$ follows from (a) which shows that $|\mathcal{D}_1(n)|/\Sigma_n = *$ and since $\mathcal{D}_1(n) = \mathcal{E}_{1,r}(n)$ if $r + 1 > n/p$. For $d \ge 2$ we will prove the result by induction on the pairs $(d, r)$ ordered lexicographically, i.e. $(d', r') < (d, r)$ if $d' < d$ or if $d' = d$ and $r' < r$. Assume inductively that the results holds for all $(r', d') < (d, r)$ and all $n$ for some $(d, r)$. We will prove it for $(d, r)$ for all $n$. If $r = 0$ then by part (b), $|\mathcal{E}_{d,0}(n)|$ is $\Sigma_n$-equivalent to $|\mathcal{D}_{d-1}(n)|$ whose orbit space is acyclic by (a) if $d = 2$ and by the induction hypothesis if $d \ge 3$ because $\mathcal{D}_{d-1}(n) = \mathcal{E}_{d-1,s}(n)$ for $s$ large enough.

We now assume that $r \ge 1$ (and $d \ge 2$). If $rp^d > n$ then $\mathcal{E}_{d,r}(n) = \mathcal{E}_{d,r-1}(n)$ and we can use the induction hypothesis directly. If $rp^d \le n$, then $|\mathcal{B}(d,r)|/\Sigma_{n'} \simeq |\mathcal{B}_0(d,r)|/\Sigma_{n'} \simeq \mathrm{pt}$ by Proposition 5.7(i), where $n' = rp^d$. We can now apply Propositions 5.8 and 5.7(i) to deduce that

$$H_*(|\mathcal{E}_{d,r}(n)|/\Sigma_n, |\mathcal{E}_{d,r-1}(n)|/\Sigma_n) = 0.$$

From the induction hypothesis on $\mathcal{E}_{d,r-1}(n)$ we conclude that $|\mathcal{E}_{d,r}(n)|/\Sigma_n$ is acyclic which completes the induction step.
(c) We obtain the following calculation, where the first isomorphism follows from Propositions 5.8, 2.7 and 2.10 and the second from Proposition 2.9, Proposition 5.7(i), the acyclicity of $|\mathcal{E}_{d,0}(m)|/\Sigma_m$ and Lemma 2.2.

$$\begin{aligned}
&H^*_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|, \mathcal{H}^1_{\Sigma_n}(k^\times)) \cong \\
&H^*_{\Sigma_{n'} \times \Sigma_m}(|\mathcal{B}(d,r) \times \mathcal{E}_{d,0}(m)|, |\mathcal{B}_0(d,r) \times \mathcal{E}_{d,0}(m)|; \mathcal{H}^1_{\Sigma_{n'}}(k^\times) \oplus \mathcal{H}^1_{\Sigma_m}(k^\times)) \cong \\
&H^*_{\Sigma_m}(|\mathcal{E}_{d,0}(m)|, |\mathcal{E}_{d,0}(m)|; \mathcal{H}^1_{\Sigma_m}(k^\times)) \oplus H^*_{\Sigma_{n'}}(|\mathcal{B}(d,r)|, |\mathcal{B}_0(d,r)|; \mathcal{H}^1_{\Sigma_{n'}}(k^\times)).
\end{aligned}$$

By Proposition 5.7(ii) these groups vanish for $* \ge 2$. $\square$

## 6. Proof of Theorem 1.1 for the fusion systems of $A_n$ and $p > 2$.

Let us first outline the proof. Since $p > 2$, $S_p(A_n) = S_p(\Sigma_n)$ (Definition 3.2) so the collection $\mathcal{D}(n)$ defined in 5.1 is a $\Sigma_n$-subposet of $S_p(A_n)$.

**Proposition 6.1.** *If $p > 2$ then $S_p^{rc}(A_n) \subseteq \mathcal{D}(n) \subseteq S_p^c(A_n)$. If $p \mid n$ then every $P \in \mathcal{D}(n)$ is centric in $A_n$, namely $C_{A_n}(P) = Z(P)$.*

Since $\mathcal{F}_S(A_n) = \mathcal{F}_S(A_m)$ if $0 \leq n - m < p$ and $p \mid m$, it is enough to prove, by Corollary 3.4, that if $p \mid n$ then

$$H^*_{A_n}(|\mathcal{D}(n)|\,;\,\mathcal{H}^1_{A_n}(k^\times)\,) = 0 \qquad \text{for all } * \geq 2.$$

Recall the definition of $\mathcal{H}^1_{\Sigma_n|A_n}(k^\times)$ from 2.11 and note that $\mathcal{D}(n)$ is by construction a $\Sigma_n$-poset. The key observation is:

**Proposition 6.2.** $H^*_{A_n}(|\mathcal{D}(n)|\,;\,\mathcal{H}^1_{A_n}(k^\times)\,) \cong H^*_{\Sigma_n}(|\mathcal{D}(n)|\,;\,\mathcal{H}^1_{\Sigma_n|A_n}(k^\times)\,)$.

Therefore the proof of Theorem 1.1 for the fusion systems of $A_n$ and for $p > 2$ follows once we show that $H^{*\geq 2}_{\Sigma_n}(|\mathcal{D}(n)|;\mathcal{H}^1_{\Sigma_n|A_n}(k^\times)\,) = 0$. In light of the fact that $\mathcal{D}(n) = \mathcal{E}_{d,r}(n)$ provided $d > \log_p n$, see Definition 5.3, the goal of this section is to prove the next result.

**Proposition 6.3.** *Fix $d \geq 1$, $r \geq 0$ and $n \geq 0$ such that $p \mid n$. Then*

$$H^{*\geq 2}_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|\,;\,\mathcal{H}^1_{\Sigma_n|A_n}(k^\times)\,) = 0.$$

We will now fill in the details.

***Proof of Proposition 6.1.*** We have remarked that $S_p(\Sigma_n) = S_p(A_n)$ (Definition 3.2). Note that $S_p^c(A_n) = S_p^c(\Sigma_n)$ because $p > 2$ and $|C_{\Sigma_n}(P)\colon C_{A_n}(P)| \leq 2$. Also $S_p^r(A_n) = S_p^r(\Sigma_n)$ by Proposition D.1. The result follows from Proposition 5.2. $\square$

***Proof of Proposition 6.2.*** Set $X = |\mathcal{D}(n)|$ as a $\Sigma_n$-space. Proposition 2.7 and Definition 2.11 show that $C^*_{A_n}(X{\downarrow}^{\Sigma_n}_{A_n}\,;\,\mathcal{H}^1_{A_n}(k^\times)\,) \cong C^*_{\Sigma_n}(X\,;\,\mathcal{H}^1_{A_n}(k^\times){\uparrow}^{\Sigma_n}_{A_n})$. The result follows since $\mathcal{H}^1_{A_n}(k^\times){\uparrow}^{\Sigma_n}_{A_n} = \mathcal{H}^1_{\Sigma_n|A_n}(k^\times)$. $\square$

**Definition 6.4.** Recall from Definition 2.11 that there is a natural transformation of coefficient functors $\mathcal{H}_{\Sigma_n} \to \mathcal{H}_{\Sigma_n|A_n}$. Let $\mathcal{N}$ and $\mathcal{M}$ denote the kernel and cokernel of this natural transformation. Let $\mathcal{M}'$ denote the image of this natural transformation. There are short exact sequences

$$0 \to \mathcal{N} \to \mathcal{H}^1_{\Sigma_n}(k^\times) \xrightarrow{\alpha} \mathcal{M}' \to 0 \quad \text{and} \quad 0 \to \mathcal{M}' \to \mathcal{H}^1_{\Sigma_n|A_n}(k^\times) \to \mathcal{M} \to 0.$$

**Definition 6.5.** Let $K$ be a subgroup of $\Sigma_n$. Denote $K \cap A_n$ by $K^{\mathrm{ev}}$. Let $K^{\mathrm{odd}}$ denote the set of the odd permutations in $K$.

**Proposition 6.6.** *Let $X \cong \Sigma_n/K$ be a transitive $\Sigma_n$-set where $K \leq \Sigma_n$ contains odd permutations. Then $\mathcal{H}^1_{\Sigma_n}(k^\times)(X) \to \mathcal{H}^1_{\Sigma_n|A_n}(k^\times)(X)$ has the form $\mathrm{Hom}(K, k^\times) \to \mathrm{Hom}(K^{\mathrm{ev}}, k^\times)$ induced by the inclusion $K^{\mathrm{ev}} \leq K$.*

*Proof.* $A_n$ acts transitively on $X$ with isotropy $K^{\mathrm{ev}}$ because $KA_n = \Sigma_n$. The result follows by the natural identification $\mathcal{H}^1_{\Sigma_n}(k^\times)(X) = \mathrm{Hom}(K, k^\times)$ and $\mathcal{H}^1_{\Sigma_n|A_n}(k^\times)(X) = \mathrm{Hom}(K^{\mathrm{ev}}, k^\times)$, see e.g. Proposition B.1(1). $\square$

**Lemma 6.7.** *If $P_0 \leq \cdots \leq P_k$ is a $k$-simplex in $\mathcal{D}(n)$ where $p$ is odd, see Definition 5.1, then $N_{\Sigma_n}(P_\bullet)$ contains an odd permutation.*

*Proof.* **Step 1:** Reduction to the case $n = p^d$ and $P_k = V_{c_1,\dots,c_r}$ is a basic subgroup of degree $d$. Since the groups $P_i$ are products of basic subgroups, by looking at the orbits of $P_k$ we see that $P_\bullet = P_\bullet^{(1)} \times \cdots \times P_\bullet^{(n)}$ where every $P_\bullet^{(j)}$ is a $k$-simplex with $P_k^{(j)}$ a basic subgroup of degree $d_j$. Now, $N_{\Sigma_n}(P_\bullet) \supseteq \prod_{j=1}^n N_{\Sigma_{p^{d_j}}}(P_\bullet^{(j)})$ and we may therefore assume that $P_k$ is a basic subgroup $V_{c_1,\dots,c_r}$ of degree $d$ in $\Sigma_n$ where $n = p^d$.
**Step 2:** Completion of the proof. We will denote $\mathbf{c} = (c_1, \dots, c_r)$. Since $p$ is odd, Proposition 4.5(iii) implies that the diagonal copy of $\mathrm{GL}_{c_1}(p)$ in $\mathrm{GL}_{c_1}(p)^{p^{c_2+\cdots+c_r}} \leq \mathrm{GL}_{\mathbf{c}}(p)$ contains odd permutations. Therefore it only remain to prove that $\mathrm{GL}_{c_1}(p)$ normalises a $P_k$-conjugate of $P_\bullet$.

We prove the last claim by induction on $d$. If $d = 1$ then $P_i = V_1$ for all $i$ so there is nothing to prove. Assume that $d \geq 2$. We will prove the claim by induction on $k$. When $k = 0$ there is nothing to prove. So we assume that $k \geq 1$.

If $P_{k-1}$ is basic of degree $d$ then by Lemma E.1(b) we may assume, up to conjugation by an element of $P_k$, that $P_{k-1} = V_{\mathbf{e}}$ where $\mathbf{c}$ refines $\mathbf{e} = (e_1, \dots, e_m)$, see Definition 4.3. By induction hypothesis on $k-1$, up to conjugation by an element of $P_{k-1}$, we may assume that $\mathrm{GL}_{e_1}(p)$ normalises $P_0 \leq \cdots \leq P_{k-1}$ and therefore $\mathrm{GL}_{c_1}(p) \leq \mathrm{GL}_{e_1}(p)$ normalises $P_\bullet$.

We now assume that $P_{k-1}$ is a product of basic subgroups of degrees $\leq d-1$. Set $P'_k = \delta_{p^{d-1}}(P_k)$, see Definition 4.7. It is clear from Lemmas 4.8 and 4.9 that $\mathrm{GL}_{c_1}(p)$ normalises $P'_k$ and that $P'_k$ contains $P_{k-1}$ and that $P'_k = (V_{c_1,\dots,c_{r-1}})^{p^{c_r}}$. By looking at the orbits of $P'_k$ we see that the $k$-simplex $P_0 \leq \cdots \leq P_{k-1} \leq P'_k$ which we denote $P'_\bullet$, is a product of $p^{c_r}$ simplices of the form $Q_\bullet^{(i)}$ where $Q_k^{(i)} = V_{c_1,\dots,c_{r-1}}$. By induction hypothesis on $d-1$, up to conjugation by an element of $Q_k^{(i)}$ we may assume that $\mathrm{GL}_{c_1}(p) \leq \mathrm{GL}_{c_1,\dots,c_{r-1}}(p)$ normalises $Q_\bullet^{(i)}$. Doing it for each factor $Q_\bullet^{(i)}$ of $P'_\bullet$ we see that $\mathrm{GL}_{c_1}(p)$ normalises, via its diagonal inclusion in $\mathrm{GL}_{c_1,\dots,c_{r-1}}(p)^{p^{c_r}}$, a conjugate of $P_\bullet$ by an element of $P'_k \subseteq P_k$. This completes the induction step for $k$ and therefore for $d$. □

**Proposition 6.8.** *Recall the coefficient functors $\mathcal{M}$ and $\mathcal{N}$ from 6.4 and fix $d \geq 2$, $r \geq 1$ and $n \geq 0$ such that $p \mid n$ and $rp^d \leq n$. Then*

$$H^*_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathcal{N}) = 0$$
$$H^*_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathcal{M}) = 0$$

*Proof.* Consider a $k$-simplex $P_\bullet$ in the relative space $|\mathcal{E}_{d,r}(n)| - |\mathcal{E}_{d,r-1}(n)|$. Our goal now is to calculate $\mathcal{N}([P_\bullet])$ and $\mathcal{M}([P_\bullet])$.

By Proposition 5.8, $P_\bullet$ is conjugate to $P'_\bullet \times P''_\bullet$ where $P'_\bullet$ is a simplex in the relative space $(|\mathcal{B}(d,r)|, |\mathcal{B}^0(d,r)|)$ and $P''_\bullet \in |\mathcal{E}_{d,0}(m)|$ where $m = n - rp^d$.

Moreover

$$N_{\Sigma_n}(P_\bullet) = N_{\Sigma_{rp^d}}(P'_\bullet) \times N_{\Sigma_m}(P''_\bullet) = (\prod_{i=1}^{t} N_{\Sigma_{p^d}}(Q^{(i)}_\bullet) \wr \Sigma_{e_i}) \times N_{\Sigma_m}(P''_\bullet).$$

**Case I:** $t > 1$, namely $P'_\bullet$ is a product of at least two non-conjugate simplices in $|\mathcal{B}(d)|$. In this case Lemmas 6.7 and A.6(b) and Proposition 6.6 imply that $\mathcal{N}([P_\bullet]) = C_2$ and $\mathcal{M}([P_\bullet]) = 0$. Also, $\mathcal{N}([P_\bullet]) \to \mathcal{H}^1_{\Sigma_n}(k^\times)([P_\bullet])$ is $\mathrm{Hom}(-, k^\times)$ applied to the signature map $\mathrm{Iso}_{\Sigma_n}(P_\bullet) \le \Sigma_n \to C_2$.
**Case II:** $t = 1$ and $m > 0$. Note that $m \ge p$ since $p \mid n$ so the simplex $P''_\bullet$ appears in the product. Lemma 6.7 applies to $P'_\bullet$ and $P''_\bullet$, whence Lemma A.6(b) and Proposition 6.6 show that $\mathcal{M}([P_\bullet]) = 0$ and that $\mathcal{N}([P_\bullet]) = C_2$ maps to $\mathcal{H}^1_{\Sigma_n}(k^\times)([P_\bullet])$ via $\mathrm{Hom}(-k^\times)$ applied to the signature map $\mathrm{Iso}_{\Sigma_n}(P_\bullet) \to C_2$.
**Case III:** $t = 1, m = 0$ and $r = 1$. Note that $P_\bullet = P'_\bullet$ and by Proposition 4.5 we may assume that $P_0 = V_d$ and $P_k = V_{c_1,\dots,c_t}$ and also $N_{\Sigma_n}(P_\bullet) = Q \rtimes \mathrm{GL}_{\mathbf{c}}(p)$ for some $Q \le P_k$. By Lemma A.9, $O^{p'}(N_{\Sigma_n}(P_\bullet)) \supseteq \Gamma^2(N_{\Sigma_n}(P_\bullet))$. Proposition 6.6 and Lemmas 6.7 and A.6(a) imply that $\mathcal{M}([P_\bullet]) = 0$ and $\mathcal{N}([P_\bullet]) = C_2$ which maps to $\mathcal{H}^1_{\Sigma_n}(k^\times)([P_\bullet])$ via $\mathrm{Hom}(-, k^\times)$ applied to the signature map $\mathrm{Iso}_{\Sigma_n}(P_\bullet) \to C_2$.
**Case IV:** $t = 1, m = 0$ and $r \ge 3$. This time we apply Lemmas 6.7 and A.6(c) to deduce that $\mathcal{M}([P_\bullet]) = 0$ and $\mathcal{N}([P_\bullet]) = C_2$ as before.
**Case V:** $t = 1, m = 0$ and $r = 2$. In this case Lemmas A.7 and 6.7 show that $\mathcal{M}([P_\bullet]) = C_2$ and $\mathcal{N}([P_\bullet]) = C_2$.

We now deduce that $\mathcal{N}([P_\bullet]) = C_2$ for all the simplices in the relative $\Sigma_n$-complex $(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|)$. The maps $\mathcal{N}([P_\bullet]) \to \mathcal{N}([Q_\bullet])$ where $Q_\bullet$ is a face of $P_\bullet$ are the identity because $\mathcal{N}([P_\bullet])$ is obtained by applying $\mathrm{Hom}(-, k^\times)$ to the signature map $\mathrm{Iso}_{\Sigma_n}(P_\bullet) \to C_2$. The acyclicity of $|\mathcal{E}_{d,r}(n)|/\Sigma_n$, see Proposition 5.4, and Proposition 2.3 imply the first statement of this proposition, namely

$$\begin{aligned} H^*_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathcal{N}) &\cong H^*_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathrm{Const}_{C_2}) \\ &\cong H^*(|\mathcal{E}_{d,r}(n)|/\Sigma_n, |\mathcal{E}_{d,r-1}(n)|/\Sigma_n; C_2) = 0. \end{aligned}$$

We also see that $\mathcal{M}$ vanishes on all the simplices $P_\bullet$ of $(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|)$ except those in Case V, i.e. if $n = 2p^d$ and $r = 2$. In this case, $P_\bullet = Q_\bullet \times Q_\bullet$ where $Q_\bullet$ is a $k$-simplices in $|\mathcal{B}(d)|$. Since $Q_0 = V_d$ we deduce that $Q_\bullet$ is a $k$-simplex in the relative space $(|\mathcal{B}(d)|, |\mathcal{B}_0(d)|)$. Thus, $\mathcal{M}$ vanishes on all the simplices of $(|\mathcal{E}_{d,2}(n)|, |\mathcal{E}_{d,1}(n)|)$ except those which are conjugate to $Q_\bullet \times Q_\bullet$ and on the orbit of these simplices it has constant value $C_2$ by Lemma A.7. Together with Propositions 5.7(i) and 2.3 we now see that

$$H^*_{\Sigma_n}(|\mathcal{E}_{d,2}(n)|, |\mathcal{E}_{d,1}(n)|; \mathcal{M}) \cong H^*_{\Sigma_{p^d}}(|\mathcal{B}(d)|, |\mathcal{B}_0(d)|; \mathrm{Const}_{C_2}) = 0.$$

We have thus seen that $H^*_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathcal{M}) = 0$ for any $n$. □

***Proof of Proposition 6.3.*** We use induction on the pairs $(d, r)$ ordered lexicographically. If $d = 1$ and $n - rp \ge p$ then $\mathcal{E}_{1,r}(n) = \emptyset$ and the result is trivial. If $d = 1$ and $n - rp < p$ then $\mathcal{E}_{1,r}(n) = \mathcal{D}_1(n)$ and the result follows from Proposition 5.4(a). We therefore assume that $d \geq 2$. If $r = 0$ then the induction step follows from Proposition 5.4(b) and the induction hypothesis. If $r > n/p^d$ it follows directly from the induction hypothesis since $\mathcal{E}_{d,r}(d) = \mathcal{E}_{d,r-1}(n)$. If $1 \leq r \leq n/p^d$ then Propositions 6.8 and 5.4(c) imply that

$$H^*_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathcal{H}^1_{\Sigma_n|A_n}(k^\times)) \cong \\ \cong H^*_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathcal{H}^1_{\Sigma_n}(k^\times)) = 0$$

for all $* \geq 2$. The induction step follows from the induction hypothesis on $(d, r-1)$ and the long exact sequence in cohomology. □

## 7. Proof of Theorem 1.1 for the fusion system of $A_n$ and $p = 2$.

Throughout this section $p = 2$. First we note that the fusion system of $A_n$ is isomorphic to that of $A_m$ where $p \mid m$ and $0 \leq n - m < p$. We will therefore assume throughout this section that $p \mid n$, namely that $n$ is even.

Let us outline the proof. Recall from 3.2 the definition of the posets $S^r_p(G)$, $S^c_p(G)$ and $S^{rc}_p(G)$.

**Proposition 7.1.** *If $n$ is even then every $P \in S^{rc}_p(A_n)$ is centric (not just $p$-centric) in $A_n$, namely $C_{A_n}(P) = Z(P)$. In fact, $C_{\Sigma_n}(P) \leq P$.*

Observe that $\Sigma_n$ acts on $S_p(\mathcal{A}_n), S^c_p(A_n), S^r_p(A_n)$ and $S^{rc}_p(A_n)$ by conjugation because $A_n \lhd \Sigma_n$, thus the $A_n$-space $|S^{rc}_p(A_n)|$ can be written $|S^{rc}_p(A_n)| \downarrow^{\Sigma_n}_{A_n}$.

**Proposition 7.2.** *There is a $\Sigma_n$-homotopy equivalence $|S^r_p(A_n)| \to |S^r_p(\Sigma_n)|$ which restricts to a $\Sigma_n$-homotopy equivalence $|S^{rc}_p(A_n)| \to |S^{rc}_p(\Sigma_n)|$.*

Recall from Proposition 5.2 that $S^{rc}_p(\Sigma_n) \subseteq \mathcal{D}(n) \subseteq S^c_p(\Sigma_n)$. Hence, Bouc's result [2, Proposition 6.6.5] implies that $|S^{rc}_p(\Sigma_n)| \subseteq |\mathcal{D}(n)|$ is a $\Sigma_n$-equivalence. Set $\mathcal{M} := \mathcal{H}^1_{A_n}(k^\times)\uparrow^{\Sigma_n}_{A_n}$ (Definitions 2.6 and 2.1). By Proposition 7.2, see also Definition 2.6, we obtain isomorphisms

$$H^*_{A_n}(|S^{rc}_p(A_n)|; \mathcal{H}^1_{A_n}(k^\times)) \cong H^*_{\Sigma_n}(|S^{rc}_p(A_n)|; \mathcal{M}) \cong H^*_{\Sigma_n}(|\mathcal{D}(n)|; \mathcal{M}).$$

By Corollary 3.4 it remains to prove that these groups vanish for $* = 2$. Since $\mathcal{D}(n) = \mathcal{E}_{d,r}(n)$ if $p^d > n$, see Definition 5.3, the goal of this section is therefore:

**Proposition 7.3.** *Fix $p = 2$ and $d \geq 1$ and $r \geq 0$ and an even $n \geq 0$. Then for $\mathcal{M} = \mathcal{H}^1_{A_n}(k^\times)\uparrow^{\Sigma_n}_{A_n}$*

$$H^{*\geq 1}_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|; \mathcal{M}) = 0.$$

We now fill in the details. The proof of Propositions 7.1 and 7.2, which are probably known to some group theorists, is deferred to Appendix D.

**Lemma 7.4.** *Fix $p = 2$. Let $P_0 \leq \cdots \leq P_q$ be an inclusion of basic $p$-subgroups of degree $d$ in $\Sigma_{p^d}$ and set $N = \cap_{i=0}^q N_{\Sigma_{p^d}}(P_i)$. Then for every $m \geq 1$,*

$$\mathrm{Hom}(N \wr \Sigma_m, k^\times) = 0$$

*where $k$ is an algebraically closed field of characteristic 2.*

*Proof.* By Proposition 4.5, $N = Q \rtimes \mathrm{GL}_{\mathbf{e}}(p)$ for some $Q \leq P_q$. By Lemma A.9, $\mathrm{Hom}(\mathrm{GL}_{\mathbf{e}}(p), k^\times) = (\mathbb{F}_2^\times)^{\mathbf{e}} = 0$ and since $k^\times$ has no 2-torsion, then also $\mathrm{Hom}(\Sigma_m, k^\times) = 0$. □

***Proof of Proposition 7.3.*** We will use induction on the pairs $(d, r)$ ordered lexicographically. Thus, we fix $(d, r)$ and assume that the results holds for $\mathcal{E}_{d',r'}(n)$ where $n$ is even and $(d', r') < (d, r)$, namely $d' < d$ or $d' = d$ and $r' < r$.

If $d = 1$ and $n - rp \geq p$ then $\mathcal{E}_{1,r}(n) = \emptyset$ and the result is trivial. If $d = 1$ and $r - rp < p$ then $\mathcal{E}_{1,r}(n) = \mathcal{D}_1(n)$ and the result follows from Proposition 5.4(a). For $d \geq 2$ the induction step follows from Proposition 5.4(b) if $r = 0$ and directly from the induction hypothesis if $r > n/p^d$ because $\mathcal{E}_{d,r}(n) = \mathcal{E}_{d,r-1}(n)$. If $d \geq 2$ and $0 < r \leq n/p^d$ the induction step for $(d, r)$ will follow from the induction hypothesis on $\mathcal{E}_{d,r-1}(n)$ and the long exact sequence in cohomology if we prove that

$$H^{*\geq 1}_{\Sigma_n}(|\mathcal{E}_{d,r}(n)|, |\mathcal{E}_{d,r-1}(n)|; \mathcal{M}) = 0.$$

This will be our goal for the remainder of the proof. By Propositions 5.4(b) and 5.8, see also Definition 2.6, we need to prove that

$$(\dagger)\quad H^{*\geq 1}_{\Sigma_{n'} \times \Sigma_m}(|\mathcal{B}(d, r)| \times |\mathcal{D}_{d-1}(m)|, |\mathcal{B}_0(d, r)| \times |\mathcal{D}_{d-1}(m)|; \mathcal{M} \downarrow^{\Sigma_n}_{\Sigma_{n'} \times \Sigma_m}) = 0$$

where $n' = rp^d$ and $m = n - n'$. Moreover, the isotropy group of a $t$-simplex $P_\bullet$ in the relative space $(|\mathcal{B}(d, r)| \times |\mathcal{D}_{d-1}(m)|, |\mathcal{B}_0(d, r)| \times |\mathcal{D}_{d-1}(m)|)$ has the form

$$(\ddagger)\qquad \mathrm{Iso}_{\Sigma_n}(P_\bullet) = \mathrm{Iso}_{\Sigma_{n'} \times \Sigma_m}(Q_\bullet \times R_\bullet) = (\prod_{i=1}^s N(Q_\bullet^{(i)}) \wr \Sigma_{e_i}) \times \mathrm{Iso}_{\Sigma_m}(R_\bullet)$$

where $Q_\bullet^{(1)}, \ldots, Q_\bullet^{(s)}$ are non-conjugate $t$-simplices in $|\mathcal{B}(d)|$ and $Q_0^{(i)} = V_d$. Also $R_\bullet$ is suppressed if $m = 0$ (note that $m$ is even).

Let us first assume that $d = 2$. We will show that $\mathcal{M}\downarrow^{\Sigma_n}_{\Sigma_{n'} \times \Sigma_m}$ vanishes on all the simplices $P_\bullet$ except, possibly the 0-simplices and their degeneracies. This clearly implies (†).

Since $R_\bullet \in |\mathcal{D}_1(m)|$, we may assume that $R_\bullet = (V_1)^{m/2}$, whence, by Propositions 5.5 and 4.5, $\mathrm{Iso}_{\Sigma_m}(R_\bullet) = V_1 \wr \Sigma_{m/2}$. This group always contains odd permutations unless $m = 0$.

By Proposition 4.5 we see that $Q_\bullet^{(i)}$ has either the form $V_2 \leq \cdots \leq V_2$ or $V_2 \leq \ldots V_2 < V_{1,1} \cdots \leq V_{1,1}$. Thus, either $N(Q_\bullet^{(i)}) = N(V_2) = V_2 \rtimes \Sigma_3$ or $N(Q_\bullet^{(i)}) = N(V_{1,1}) \cap N(V_2) = V_{1,1}$ (use Proposition 4.5 and note that $p = 2$).

Therefore all the factors in (‡) contain odd permutations. We conclude from Propositions 7.4 and 6.6 and Lemma A.4 that if either $s \geq 2$ or $m \geq 2$ then

$$\mathcal{M}{\downarrow}^{\Sigma_n}_{\Sigma_{n'} \times \Sigma_m}([P_\bullet]) = \mathrm{Hom}(\mathrm{Iso}(Q_\bullet \times R_\bullet) \cap A_n, k^\times) = \mathrm{Hom}(\mathrm{Iso}(Q_\bullet \times R_\bullet), k^\times) = 0.$$

If $m = 0$ and $s = 1$ then $P_\bullet = (Q_\bullet^{(1)})^r$ and therefore $\mathrm{Iso}(P_\bullet)$ is either $V_{1,1} \wr \Sigma_r$ if $Q_t^{(1)} = V_{1,1}$, or $N(V_2) \wr \Sigma_r$ if $Q_t^{(1)} = V_2$. Note that $\Sigma_r$ acts by permuting the orbits of $(V_2)^r$ and it is therefore contained in $A_n$. In the first case we note that $(V_{1,1} \wr \Sigma_r) \cap A_n = ((V_{1,1})^r \cap A_n) \rtimes \Sigma_r$ and since $k^\times$ has no 2-torsion

$$\mathcal{M}\downarrow^{\Sigma_n}_{\Sigma_{n'} \times \Sigma_m}([P_\bullet]) = \mathrm{Hom}(((V_{1,1})^r \cap A_n) \rtimes \Sigma_r, k^\times) = \mathrm{Hom}(\Sigma_r, k^\times) = 0.$$

In the second case, namely if $\mathrm{Iso}(P_\bullet) = N(V_2) \wr \Sigma_r$, then $Q_\bullet^{(1)}$ has the form $V_2 \leq \cdots \leq V_2$ and therefore $P_\bullet$ is a degeneracy of a 0-simplex in $|\mathcal{B}(d,r)|$. We therefore see that $\mathcal{M}\downarrow^{\Sigma_n}_{\Sigma_{n'} \times \Sigma_m}$ vanishes on all the simplices $P_\bullet$ except, possibly, on certain 0-simplices and their degeneracies.

Finally, let us assume that $d \geq 3$. Lemma D.10 implies that the product of the first $s$ factors in (‡) is contained in $A_{n'}$ because $Q_0^{(i)} = V_d$ for all $i = 1, \ldots, s$. It follows that $\mathrm{Iso}(P_\bullet) \leq A_{n'} \times \Sigma_m$. We now claim that $\mathcal{M}{\downarrow}^{\Sigma_n}_{\Sigma_{n'} \times \Sigma_m}$ and the coefficient functor $\mathcal{N} = 0_{\Sigma_{n'}} \oplus (\mathcal{H}^1_{A_m}(k^\times){\uparrow}^{\Sigma_m}_{A_m})$, see Definition 2.8, have the same values on $P_\bullet$. Indeed, by Lemma 7.4 and Proposition B.1,

$$\mathcal{M}\downarrow^{\Sigma_n}_{\Sigma_{n'} \times \Sigma_m}([Q_\bullet \times R_\bullet]) = \bigoplus_{[\Sigma_n : A_n \cdot \mathrm{Iso}(R_\bullet)]} \mathrm{Hom}(\mathrm{Iso}(Q_\bullet) \times (\mathrm{Iso}(R_\bullet) \cap A_m), k^\times) =$$
$$\bigoplus_{[\Sigma_m : A_m \cdot \mathrm{Iso}(R_\bullet)]} \mathrm{Hom}(\mathrm{Iso}(R_\bullet) \cap A_m, k^\times) = \big(0_{\Sigma_{n'}}([Q_\bullet])\big) \oplus \big(\mathcal{H}_{A_m}\uparrow^{\Sigma_m}_{A_m}([R_\bullet])\big).$$

From Propositions 2.9 we get

$$H^*_{\Sigma_{n'} \times \Sigma_m}(|\mathcal{B}(d,r)| \times |\mathcal{D}_{d-1}(m)|, |\mathcal{B}_0(d,r)| \times |\mathcal{D}_{d-1}(m)|; 0_{\Sigma_{n'}} \oplus \mathcal{H}^1_{A_m}(k^\times){\uparrow}^{\Sigma_m}_{A_m}) \cong$$
$$H^*_{\Sigma_m}(|\mathcal{D}_{d-1}(m)|, |\mathcal{D}_{d-1}(m)|; \mathcal{H}^1_{A_m}(k^\times){\uparrow}^{\Sigma_m}_{A_m}) = 0$$

This establishes (†) and completes the proof. □

## 8. Proof of Theorem 1.1 for $\mathbf{GL}_d(p^r)$ and $\mathbf{SL}_d(p^r)$ at $p$

Set $q = p^r$ and $\mathbb{F} = \mathbb{F}_q$. We also fix an algebraically closed field $k$ of characteristic $p$. Set $G = \mathrm{GL}_d(q)$ and $K = \mathrm{SL}_d(q)$. We also fix a Sylow $p$-subgroup $S \leq G$. It is also a Sylow for $K$. Let $\mathcal{F}_G$ and $\mathcal{F}_K$ denote the fusion systems of $G$ and $K$ defined on $S$ . Let $\mathcal{L}_G$ and $\mathcal{L}_K$ be the associated centric linking systems. Clearly, Theorem 1.1 is trivial if $d = 1$. We therefore assume **throughout this section that** $d \geq 2$.

Set $\mathcal{P}art_0(d) \overset{\mathrm{def}}{=} \mathcal{P}art(d) - \{(d)\}$, see Definition 4.3. For every $\mathbf{c} = (c_1, \ldots, c_t)$ in $\mathcal{P}art_0(d)$ we set $\gcd(\mathbf{c}) = \gcd(c_1, \ldots, c_t)$. For any $n \geq 1$ we write $n \mid \mathbf{c}$ if $n$ divides $c_1, \ldots, c_t$. Observe that if $n \mid \mathbf{c}$ then $n \mid d$ since $d = c_1 + \cdots + c_t$.

The proof of Theorem 1.1 for $G$ and $K$ relies on the next two propositions which we will prove later.

For every $\mathbf{c} = (c_1, \dots, c_t)$ in $\mathcal{P}art_0(d)$ define the following subgroups of $\mathrm{GL}_d(q)$ and $\mathrm{SL}_d(q)$,

$$\mathrm{GL}_{\mathbf{c}}(q) = \mathrm{GL}_{c_1}(q) \times \cdots \times \mathrm{GL}_{c_t}(q) \qquad \text{and} \qquad \mathrm{SL}_{\mathbf{c}}(q) = \mathrm{GL}_{\mathbf{c}}(q) \cap \mathrm{SL}_d(q).$$

Clearly $Z(G)$, namely the group of scalar matrices, is contained in $\mathrm{GL}_{\mathbf{c}}(q)$. Also if $\mathbf{c} \le \mathbf{c}'$ then $\mathrm{GL}_{\mathbf{c}'}(q) \le \mathrm{GL}_{\mathbf{c}}(q)$ and $\mathrm{SL}_{\mathbf{c}'}(q) \le \mathrm{SL}_{\mathbf{c}}(q)$. Set $Z = Z(G)$.

**Proposition 8.1.** *Fix a subgroup $Z' \le Z$. Consider the functor $F\colon \mathcal{P}art_0(d) \to$ **Ab** defined by $F\colon \mathbf{c} \mapsto \mathrm{Hom}(\mathrm{GL}_{\mathbf{c}}(q)/Z', k^\times)$. Then $H^{*\ge 2}(\mathcal{P}art_0(d); F) = 0$.*

**Proposition 8.2.** *Fix $Z' \le Z \cap \mathrm{SL}_d(q)$. Consider the functor $F\colon \mathcal{P}art_0(d) \to$ **Ab** defined by $F\colon \mathbf{c} \mapsto \mathrm{Hom}(\mathrm{SL}_{\mathbf{c}}(q)/Z', k^\times)$. Then $H^{*\ge 2}(\mathcal{P}art_0(d); F) = 0$.*

***Proof of Theorem 1.1 for $\mathcal{F}_G$ and $\mathcal{F}_K$.*** We describe the poset $S_p^r(G)$. A quick and down-to-earth overview can be found in [2, §6.8]. Let $\mathrm{Fl}(\mathbb{F}^d)$ denote the poset of all the flags in $\mathbb{F}^d$ where a flag $\mathbf{W}$ is a chain of subspaces of $\mathbb{F}^d$ ordered by inclusion, each has dimension between 1 and $d-1$. Let $\mathrm{Fl}_0(\mathbb{F}^d)$ be the subposet of the non-trivial flags, namely flags which consist of at least one subspace. There is an isomorphism of posets $\mathrm{Fl}(\mathbb{F}^d) \to S_p^r(G)$ given by $\mathbf{W} \mapsto P := O_p(N_G(\mathbf{W}))$ where $N_G(\mathbf{W})$ is the subgroup of $G$ which leaves the flag $\mathbf{W}$ invariant. In addition, $N_G(P) = N_G(\mathbf{W})$ and therefore if $P' \le P$ then $N_G(P) \le N_G(P')$. It is easy to check that if $P \in S_p^r(G)$ then $C_G(P) = Z(P) \times Z(G)$ unless $P = 1$. It follows that $S_p^{rc}(G) = S_p^r(G) - \{1\}$ and that $C'_G(P) = Z(G)$ for $P \in S_p^{rc}(G)$. Clearly $S_p(K) = S_p(G)$ and Proposition D.1 implies that $S_p^{rc}(K) = S_p^{rc}(G)$.

By viewing $\mathcal{P}art(d)$ as partitions into intervals of the standard basis of $\mathbb{F}^d$, we obtain an order preserving map $\mathcal{P}art_0(d) \hookrightarrow \mathrm{Fl}_0(\mathbb{F}^d)$, hence a map $\mathbf{c} \mapsto P_{\mathbf{c}}$ of posets $\mathcal{P}art_0(d) \to S_p^{rc}(G)$. We note that $N_G(P_{\mathbf{c}}) = P_{\mathbf{c}} \rtimes \mathrm{GL}_{\mathbf{c}}(q)$ and therefore $N_K(P_{\mathbf{c}}) = P_{\mathbf{c}} \rtimes \mathrm{SL}_{\mathbf{c}}(q)$ since $P_{\mathbf{c}} \le K$. Observe that $|\mathcal{P}art_0(d)| \cong |\mathrm{Fl}_0(\mathbb{F}^d)|/G \cong |S_p^{rc}(G)|/G$ because $G$ acts transitively on the set of bases of $\mathbb{F}^d$. Thus, the subdivision poset $S(\mathcal{P}art_0(d))$, see Section 3, which is the poset of non-degenerate simplices of $|\mathcal{P}art_0(d)|$ is isomorphic to $[S(S_p^{rc}(G))] \overset{\text{def}}{=} S(S_p^{rc}(G))/G$.

Let $\mathcal{C}$ denote the collection of the subgroups $P$ of $S$ such that $P \in S_p^{rc}(G)$. Recall the definition of the posets $[S(\mathcal{F}_G^{\mathcal{C}})]$ and $[S(\mathcal{F}_K^{\mathcal{C}})]$ from the introduction and note that since $\mathcal{F}_G$-isomorphisms are the same thing as conjugation in $G$, $[S(\mathcal{F}_G^{\mathcal{C}})]$ is the set of non-degenerate simplices of $|S_p^{rc}(G)|/G$. We deduce that $S(\mathcal{P}art_0(d)) = [S(\mathcal{F}_G^{\mathcal{C}})]$. Also note that

$$|N_G(P_{\mathbf{c}})\colon N_K(P_{\mathbf{c}})| = q - 1 = |G\colon K|$$

for any $\mathbf{c} \in \mathcal{P}art_0(d)$ because $\mathrm{GL}_{\mathbf{c}}(q)$ contains all the diagonal matrices and therefore $|S_p^{rc}(G)|/K = |S_p^{rc}(G)|/G$. Consequently, $[S(\mathcal{F}_K^{\mathcal{C}})] = S(\mathcal{P}art_0(d))$. In addition we note that $C'_K(P_{\mathbf{c}}) = Z(G) \cap K = Z(K)$ for any $\mathbf{c} \in \mathcal{P}art_0(d)$.

Set $Z = Z(G) \cong \mathbb{F}^\times$ and $Z' = Z \cap K$. Define functors $F_G, F_K \colon \mathcal{P}art_0(d) \to \mathbf{Ab}$ by

$$F_G \colon \mathbf{c} \mapsto \mathrm{Hom}(\mathrm{GL}_{\mathbf{c}}(q)/Z, k^\times), \qquad F_K \colon \mathbf{c} \mapsto \mathrm{Hom}(\mathrm{SL}_{\mathbf{c}}(q)/Z', k^\times).$$

Recall the definition of the functor $\mathcal{A}^1_{\mathcal{L}_G} \colon [S(\mathcal{F}^{\mathcal{C}}_G)] \to \mathbf{Ab}$ from the introduction. For any $\mathbf{c}_0 \le \cdots \le \mathbf{c}_n$ in $\mathcal{P}art_0(d)$ we note that $\mathrm{Aut}_{\mathcal{L}_G}(P_{\mathbf{c}_0} \le \cdots \le P_{\mathbf{c}_n}) = \cap_i N_G(P_{\mathbf{c}_i})/Z = N_G(P_{\mathbf{c}_n})/Z$ because $C'_G(P_{\mathbf{c}_i}) = Z$. Therefore,

$$\mathcal{A}^1_{\mathcal{L}_G}(P_{\mathbf{c}_0} \le \cdots \le P_{\mathbf{c}_n}) = \mathrm{Hom}(P_{\mathbf{c}_n} \rtimes \mathrm{GL}_{\mathbf{c}_n}(q)/Z, k^\times) \cong F_G(\mathbf{c}_n).$$

Hence, we can identify $\mathcal{A}^1_{\mathcal{L}_G}$ with $\pi^*(F_G)$ where $\pi \colon S(\mathcal{P}art_0(d)) \to \mathcal{P}art_0(d)$ is the functor from Proposition 3.1. Together with Proposition 8.1 we deduce that

$$\begin{aligned} H^{*\ge 2}([S(\mathcal{F}^{\mathcal{C}}_G)]; \mathcal{A}^1_{\mathcal{L}_G}) \cong H^{*\ge 2}(S(\mathcal{P}art_0(d)); \pi^*(F_G)) \cong \\ \cong H^{*\ge 2}(\mathcal{P}art_0(d); F_G) = 0. \end{aligned}$$

By Proposition 3.3, $\mathcal{C}$ consists of $\mathcal{F}$-centric subgroups of $S$ and contains all the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups. We can now apply Theorem 1.2.

Similarly, $\mathrm{Aut}_{\mathcal{L}_K}(P_{\mathbf{c}_0} \le \cdots \le P_{\mathbf{c}_n}) = \cap_i N_K(P_{\mathbf{c}_i})/Z' = N_K(P_{\mathbf{c}_n})/Z'$ Hence

$$\mathcal{A}^1_{\mathcal{L}_K}(P_{\mathbf{c}_0} \le \cdots \le P_{\mathbf{c}_n}) = \mathrm{Hom}(P_{\mathbf{c}_n} \rtimes \mathrm{SL}_{\mathbf{c}_n}(q)/Z', k^\times) = F_K(\mathbf{c}_n).$$

Thus, $\mathcal{A}^1_{\mathcal{L}_K}$ can be identified with $\pi^*(F_K)$. Propositions 3.1 and 8.2 imply that

$$\begin{aligned} H^{*\ge 2}([S(\mathcal{F}^{\mathcal{C}}_K)]; \mathcal{A}^1_{\mathcal{L}_K}) \cong H^{*\ge 2}(S(\mathcal{P}art_0(d)); \pi^*(F_K)) \cong \\ \cong H^{*\ge 2}(\mathcal{P}art_0(d); F_K) = 0. \end{aligned}$$

Now apply Theorem 1.2. □

**Proposition 8.3.** *Fix a prime $r$. Let $F \colon \mathcal{P}art_0(d) \to \mathbf{Ab}$ be a functor with the property that if $\mathbf{c} \le \mathbf{c}'$ are objects such that $r$ does not divide $\frac{\gcd(\mathbf{c})}{\gcd(\mathbf{c}')}$, then $F(\mathbf{c}) = F(\mathbf{c}')$. Then $F$ is acyclic, namely $H^{*\ge 1}(\mathcal{P}art_0(d); F) = 0$.*

*Proof.* For any integer $n$ let $\nu_r(n)$ denote the largest power of $r$ which divides $n$. Let $\mathcal{R}(d)$ be the poset $\{r^s \preceq r^{m-1} \preceq \cdots \preceq r \preceq 1\}$ where $r^s$ is the largest power of $r$ which *properly* divides $d$ (thus, $r^s \ne d$). There is an obvious inclusion functor

$$\theta \colon \mathcal{R}(d) \to \mathcal{P}art_0(d), \qquad r^m \mapsto (r^m, r^m, \ldots, r^m).$$

There is also a projection functor

$$\kappa \colon \mathcal{P}art_0(d) \to \mathcal{R}(d), \qquad \mathbf{c} \mapsto \nu_r(\gcd(\mathbf{c})).$$

First, we claim that for any functor $\mathcal{A} \colon \mathcal{R}(d) \to \mathbf{Ab}$ we have

$$H^{*\ge 1}(\mathcal{P}art_0(d); \kappa^*\mathcal{A}) = 0.$$

For any $k \in \mathcal{R}(d)$ the over-category $(\kappa \downarrow k)$ is the subposet of $\mathcal{P}art_0(d)$ consisting of the objects $\mathbf{c}$ such that $k|\gcd(\mathbf{c})$. This subposet contains the object $\theta(k)$ as a maximum and its nerve is therefore contractible. By [3, Ch. XI, 9.2 and 7.2] or [26, Prop. 2.3.4], and since $\mathcal{R}(d)$ contains an initial object, we get $H^*(\mathcal{P}art_0(d); \kappa^*\mathcal{A}) \cong H^*(\mathcal{R}(d); \mathcal{A}) = 0$ for all $* \geq 1$.

We complete the proof by showing that $F = \kappa^*(\theta^*(F))$. Indeed, for any $\mathbf{c} \in \mathcal{P}art_0(d)$ consider $\mathbf{d} = \theta(\kappa(\mathbf{c}))$. Then $\mathbf{c} \leq \mathbf{d}$ and by construction $r$ does not divide $\frac{\gcd(\mathbf{c})}{\gcd(\mathbf{d})} = \frac{\gcd(\mathbf{c})}{\nu_r(\gcd(\mathbf{c}))}$, so by hypothesis $F(\mathbf{c}) = F(\mathbf{d}) = \kappa^*(\theta^* F)(\mathbf{c})$. $\square$

**Corollary 8.4.** *Fix a finite abelian group $L$. Then the functors $\mathcal{P}art_0(d) \to \mathbf{Ab}$ defined below are acyclic, namely $H^{*\geq 1}(\mathcal{P}art_0(d); -)$ vanishes for these functors.*

$$\mathcal{A}' \colon \mathbf{c} \mapsto \gcd(\mathbf{c})L, \qquad \mathcal{A} \colon \mathbf{c} \mapsto L, \qquad \mathcal{A}'' \colon \mathbf{c} \mapsto L/\gcd(\mathbf{c})L.$$

*Proof.* Write $L = L_{(r_1)} \oplus \cdots \oplus L_{(r_s)}$ as a direct sum of finite abelian $r_i$-groups where $r_1, \ldots, r_s$ are the primes dividing $|L|$. This gives rise to decompositions $\mathcal{A}' = \oplus_{i=1}^s \mathcal{A}'_i$ and $\mathcal{A} = \oplus_{i=1}^s \mathcal{A}_i$ and $\mathcal{A}'' = \oplus_{i=1}^s \mathcal{A}''_i$. For each $i$ the functors $\mathcal{A}'_i, \mathcal{A}_i$ and $\mathcal{A}''_i$ satisfy the hypotheses of Proposition 8.3 with respect to $r_i$. $\square$

***Proof of Proposition 8.1.*** Set $A = \mathrm{Hom}(Z', k^\times)$ and $L = \mathrm{Hom}(\mathbb{F}^\times, k^\times)$. Let $M$ be the constant functor $\mathcal{P}art_0(d) \xrightarrow{\mathbf{c} \mapsto A} \mathbf{Ab}$ and define $F' \colon \mathcal{P}art_0(d) \to \mathbf{Ab}$ by $F' \colon \mathbf{c} \mapsto \mathrm{Hom}(\mathrm{GL}_{\mathbf{c}}(q), k^\times)$. The exact sequences $1 \to Z' \to \mathrm{GL}_{\mathbf{c}}(q) \to \mathrm{GL}_{\mathbf{c}}(q)/Z' \to 1$ for every $\mathbf{c} \in \mathcal{P}art_0(d)$, give rise to an exact sequence of functors

$$0 \to F \to F' \xrightarrow{\beta} M.$$

Set $M' = \mathrm{Im}(\beta)$ and $N = \mathrm{Coker}(\beta)$. Lemma A.9 shows there is an isomorphism $F'(\mathbf{c}) \cong L^{\mathbf{c}}$ obtained by applying $\mathrm{Hom}(-, k^\times)$ to $\prod_{i=1}^t \mathrm{GL}_{c_i}(q) \xrightarrow{\prod_t \det} (\mathbb{F}^\times)^t$. Also $Z' \subseteq Z$ induces, by applying $\mathrm{Hom}(-, k^\times)$ a surjection $L \xrightarrow{x \mapsto \bar{x}} A$. Since $Z'$ is contained in the group of scalar matrices, $Z' \leq \mathrm{GL}_{\mathbf{c}}(q) \xrightarrow{\prod_t \det} (\mathbb{F}^\times)^{\mathbf{c}}$ is the map $z \mapsto (z^{c_1}, \ldots, z^{c_t})$. It follows, by applying $\mathrm{Hom}(-, k^\times)$, that (in additive notation) $F'(\mathbf{c}) \to M(\mathbf{c})$ has the form $(x_1, \ldots, x_t) \mapsto c_1\bar{x}_1 + \cdots + c_t\bar{x}_t$. Since $L \to A$ is surjective we deduce that $M'(\mathbf{c}) = \sum_{i=1}^t c_i A = \gcd(\mathbf{c})A$, hence $N$ has the form

$$N \colon \mathbf{c} \mapsto A/\gcd(\mathbf{c})A.$$

By Corollary 8.4, $H^*(\mathcal{P}art_0(d); M) \to H^*(\mathcal{P}art_0(d); N)$ is an isomorphism for $* \geq 1$ and surjective for $* = 0$. The exact sequence $0 \to M' \to M \to N \to 0$ implies that $H^{*\geq 1}(\mathcal{P}art_0(d); M') = 0$.

Finally, recall that $F'(\mathbf{c}) = L^{\mathbf{c}}$. With the notation of Definition C.4 we can identify $F' = \kappa_L$. Proposition C.5 implies that $H^{*\geq 1}(\mathcal{P}art_0(d); F') = 0$. The short exact sequence $0 \to F \to F' \to M' \to 0$ implies that $H^{*\geq 2}(\mathcal{P}art_0(d); F) = 0$. $\square$

***Proof of Proposition 8.2.*** Consider the functors

$$F \colon \mathbf{c} \mapsto \mathrm{Hom}(\mathrm{SL}_{\mathbf{c}}(q)/Z') \qquad \text{and} \qquad F' \colon \mathbf{c} \mapsto \mathrm{Hom}(\mathrm{GL}_{\mathbf{c}}(q)/Z', k^\times)$$

and the constant functor $M\colon \mathbf{c} \mapsto \operatorname{Hom}(\mathbb{F}^\times, k^\times) \overset{\text{def}}{=} L$. Since $Z'$ is contained in the kernel of $\det\colon \mathrm{GL}_d(q) \to \mathbb{F}^\times$, for every $\mathbf{c}$ we obtain a split short exact sequence

$$1 \to \mathrm{SL}_{\mathbf{c}}(q)/Z' \to \mathrm{GL}_{\mathbf{c}}(q)/Z' \xrightarrow{\det} \mathbb{F}^\times \to 0,$$

the splitting is via $\mathbb{F}^\times \xrightarrow{\lambda \mapsto \mathrm{diag}(\lambda,1,\dots,1)} \mathrm{GL}_{\mathbf{c}}(q)$. By applying the exact functor $\operatorname{Hom}(-,k^\times)$ we obtain a short exact sequence $0 \to M \to F' \to F \to 0$. By Proposition 8.1, $H^{*\geq 2}(\mathcal{P}art_0(d); F') = 0$. Also, $H^{*\geq 1}(\mathcal{P}art_0(d); M) = 0$ by Corollary 8.4. It follows that $H^{*\geq 2}(\mathcal{P}art_0(d); F) = 0$. □

# Appendices

### A. Commutators and centralizers in permutation groups

The commutator subgroup of any group $G$ is denoted $\Gamma^2(G)$. Denote $G_{\mathrm{ab}} = G/\Gamma^2(G)$. The commutator of $x, y \in G$ is denoted $[x,y]$.

For a finite group $G$, the largest normal $p$-subgroup is denoted $O_p(G)$. We let $O^{p'}(G)$ denote the subgroup of a finite group $G$ generated by all the elements of $p$-power order. It is clearly a characteristic subgroup of $G$ which is contained in the kernel of any homomorphism $G \to k^\times$ where $k$ is an algebraically closed field of characteristic $p$.

We will denote the symmetric group on a set $\Omega$ by $\mathrm{Sym}(\Omega)$ and $\mathrm{Alt}(\Omega)$ the alternating group. For any subgroup $G \leq \mathrm{Sym}(\Omega)$ denote $G^{\mathrm{ev}} = G \cap \mathrm{Alt}(\Omega)$. The group $G$ acts freely on $\Omega$ if every $\omega \in \Omega$ is fixed by the identity element of $G$ only. The support of $g \in G$ is the set of $\omega \in \Omega$ which are not fixed by $g$. More generally, $\mathrm{supp}(G)$ is the set of $\omega \in \Omega$ which $G$ does not fix.

For any permutation group $G$, namely a subgroup of $\Sigma_n$, we have the signature homomorphism $\mathrm{sign}\colon G \to \Sigma_n/A_n \cong \mathbb{Z}/2$. Thus, $\mathrm{sign}(g) = 0 \in \mathbb{Z}/2$ if and only if $g$ is an even permutation. Clearly sign is trivial if and only if $G = G^{\mathrm{ev}}$ and it factors through $G \to G_{\mathrm{ab}}$.

**Lemma A.1.** *Fix $H \leq \Sigma_m$, $K \leq \Sigma_n$ and set $G = H \wr K$ as a subgroup of $\Sigma_{mn}$. If $|\mathrm{supp}(k)| \geq r$ for all $1 \neq k \in K$ then $|\mathrm{supp}(g)| \geq rm$ for any $g \in G - H^n$. If $H$ acts transitively and freely then for any $g \in G$ both $\mathrm{supp}(g)$ and $\mathrm{fix}(g)$ are union of orbits of the base group $H^n$ and in particular their cardinality is divisible by $m$.*

*Proof.* Since $G = H^n \rtimes K$, every $g \in G$ has the form $(h_i) \cdot k$ for $k \in K$ and $(h_i)_{i=1}^n \in H^n$. Let $S_1, \dots, S_n$ be the orbits (of length $m$) of $(\Sigma_m)^n \leq \Sigma_{mn}$. Note that $k$ permutes $S_1, \dots, S_n$ and that $\mathrm{supp}(h_i) \subseteq S_i$. Therefore $\mathrm{supp}(g)$ is the union of $\cup_{j \in \mathrm{supp}(k)} S_j$ and $\cup_i \mathrm{supp}(h_i)$. In particular, if $g \notin H^n$ then $k \neq 1$ and therefore $|\mathrm{supp}(g)| \geq |\mathrm{supp}(k)| \cdot m \geq rm$. If $H$ acts freely then either $\mathrm{supp}(h_i) = S_i$ or $\mathrm{supp}(h_i) = \emptyset$ if $h_i = 1$, hence $\mathrm{supp}(g)$ is a union of a subcollection of $S_1, \dots, S_n$. □

**Lemma A.2.** *If $G = \prod_{i=1}^n G_i$ then $O_p(G) = \prod_{i=1}^n O_p(G_i)$. If $O_p(H) = 1$ and $G = H \wr K$ for some $K \leq \Sigma_n$, then $O_p(G) = 1$ if $H \neq 1$ and $O_p(G) = O_p(K)$ if $H = 1$.*

*Proof.* The first statement is straightforward. Fix $G = H \wr K$, thus $G = H^n \rtimes K$. The result is a triviality if $H = 1$. We therefore assume that $H \neq 1$. Set $P = O_p(G)$ and note that $P \cap H^n = 1$ by hypothesis on $H$. Therefore the restriction of the projection $\pi \colon G \to K$ to $P$ is injective and it follows that $H^n = \mathrm{Ker}(\pi)$ acts trivially on $P$ by conjugation, namely $P \leq C_G(H^n)$. We leave it as an easy exercise to check that if $H \neq 1$ then $C_G(H^n) \leq H^n$. It follows that $P = 1$. □

**Proposition A.3.** *Fix non-trivial subgroups $G_1, G_2, \dots, G_n$ of $\Sigma_\Omega := Sym(\Omega)$ with* disjoint *supports $S_1, S_2, \dots, S_n$. Then*

1. *$C_{\Sigma_\Omega}(G_1 \times \cdots \times G_n) = C_{\Sigma_{S_1}}(G_1) \times \cdots \times C_{\Sigma_{S_n}}(G_n) \times \Sigma_T$ where $T$ is the complement of $S_1 \cup \cdots \cup S_n$.*
2. *If $G_1, G_2$ and $G_3$ contain odd permutations then $C_{\Sigma_\Omega}((G_1 \times G_2 \times G_3)^{\mathrm{ev}}) = C_{\Sigma_\Omega}(G_1 \times G_2 \times G_3)$.*
3. *Suppose that $G_1$ and $G_2$ are transitive on $S_1$ and $S_2$ and that they contain odd permutations. Also assume that $\Omega = S_1 \cup S_2$ and that $|S_1| \geq 4$. Then $C_{\Sigma_\Omega}((G_1 \times G_2)^{\mathrm{ev}}) = C_{\Sigma_\Omega}(G_1 \times G_2)$.*

*Proof.* The proof of (1) is elementary. To prove (2), set $G = G_1 \times G_2 \times G_3$ and consider some $u \in C_{\Sigma_\Omega}(G^{\mathrm{ev}})$. We have to prove that it centralises $G_1, G_2$ and $G_3$. By symmetry, it suffices to show that $u$ centralises $G_1$.

Fix some $g \in G_1$. If $g \in G_1^{\mathrm{ev}}$ then $[g, u] = 1$ trivially since $G_1^{\mathrm{ev}} \leq G^{\mathrm{ev}}$. Assume that $g \in G_1^{\mathrm{odd}}$. In order to show that $u^{-1}gu$ is the same permutation as $g$, we have to check their effect on every $\omega \in \Omega$. For any $\omega$ there is some $j = 1, 2, 3$ such that both $\omega$ and $u(\omega)$ do *not* belong to $S_j$. Choose $h \in G_j^{\mathrm{odd}}$ and note that it must fix $\omega$ and $u(\omega)$. Now, $u$ centralises $gh \in G^{\mathrm{ev}}$ so $g(\omega) = gh(\omega) = u^{-1}ghu(\omega) = u^{-1}gu(\omega)$. Since this holds for all $\omega \in \Omega$, we see that $u$ centralises $g$.

Finally we prove (3). Set $G = G_1 \times G_2$ and fix $u \in C(G^{\mathrm{ev}})$. We have to prove that $u$ centralises both $G_1$ and $G_2$. First, we claim that $u(s_2) \in S_2$ for any $s_2 \in S_2$. If $u(s_2) = s_1 \in S_1$ then there exists some $h \in G_1^{\mathrm{ev}}$ such that $h(s_1) \neq s_1$ because $G_1^{\mathrm{ev}}$ acts on $S_1$ with either one orbit of size $|S_1| \geq 4$ or with two orbits of size $|S_1|/2 \geq 2$. Now, $u$ centralises $h \in G^{\mathrm{ev}}$ so $s_1 \neq h(s_1) = uhu^{-1}(s_1) = uh(s_2) = u(s_2) = s_1$ which is absurd. Therefore $u(S_2) = S_2$ and $u(S_1) = S_1$. For any $g_1 \in G_1$ choose $g_2 \in G_2$ such that $g_1g_2 \in G^{\mathrm{ev}}$ and observe that for any $s_1 \in S_1$ we have $g_1(s_1) = g_1g_2(s_1) = u^{-1}g_1g_2u(s_1) = u^{-1}g_1u(s_1)$. For any $s_2 \in S_2$ we clearly have $g_1(s_2) = s_2 = u^{-1}g_1u(s_2)$. This shows that $u$ centralises $G_1$ and it similarly centralises $G_2$. Thus, $C(G^{\mathrm{ev}}) \subseteq C(G)$ and the reverse inclusion is clear. □

**Lemma A.4.** *Fix $H \leq \Sigma_s$ and $K \leq \Sigma_t$ which contain odd permutations and consider $G = H \times K \leq \Sigma_{s+t}$. Then $\Gamma^2(G^{\mathrm{ev}}) = \Gamma^2(G) = \Gamma^2(H) \times \Gamma^2(K)$. In particular $0 \to {G^{\mathrm{ev}}}_{\mathrm{ab}} \to G_{\mathrm{ab}} \xrightarrow{\mathrm{sign}} C_2 \to 0$ is exact.*

*Proof.* Fix some $w \in K^{\text{odd}}$. Note that $w$ commutes with any $x, y \in H$. It follows that $[x, y] = [xw^{\text{sign}(x)}, yw^{\text{sign}(y)}] \in \Gamma^2(G^{\text{ev}})$. Therefore $\Gamma^2(G^{\text{ev}})$ contains $\Gamma^2(H)$ and similarly it contains $\Gamma^2(K)$. But $\Gamma^2(G^{\text{ev}}) \subseteq \Gamma^2(G) = \Gamma^2(H) \times \Gamma^2(K)$. $\square$

**Proposition A.5.** *Fix some $H \leq \Sigma_k$ and set $G = H \wr \Sigma_n$. Then*

(1) *The homomorphisms $\Sigma_n \leq G$ and $H \xrightarrow{\text{diag}} H^n \leq G$ induce upon abelianization an isomorphism $G_{\text{ab}} \cong H_{\text{ab}} \times (\Sigma_n)_{\text{ab}}$.*
(2) *If $n \geq 3$ and $H^{\text{odd}} \neq \emptyset$ then $\Gamma^2(G^{\text{ev}}) = \Gamma^2(G)$ and there results a short exact sequence of abelian groups $0 \to {G^{\text{ev}}}_{\text{ab}} \to G_{\text{ab}} \xrightarrow{\text{sign}} C_2 \to 0$.*
(3) *If $n = 2$ and $H^{\text{odd}} \neq \emptyset$ then abelianisation gives rise to a short exact sequence of abelian groups $0 \to C_2 \xrightarrow{i} {G^{\text{ev}}}_{\text{ab}} \to G_{\text{ab}} \xrightarrow{\text{sign}} C_2 \to 0$ where the first group is $C_2 = H/H^{\text{ev}}$ and $i$ is induced by $H \xrightarrow{h \mapsto (h, h^{-1})} G^{\text{ev}}$.*

*Proof.* We will write $\Gamma(-)$ instead of $\Gamma^2(-)$ for short. For $h \in H$ we will write $h_i$ for the image of $h$ in the $i^{\text{th}}$ factor of $H^n \leq G$. The elements of $G$ are written in the form $\mathbf{h}\sigma$ for $\mathbf{h} \in H^n$ and $\sigma \in \Sigma_n$ and $\sigma\mathbf{h}\sigma^{-1}$ is obtained from $\mathbf{h}$ by permuting the factors of $H^n$ via $\sigma$. Note that $x_i$ and $y_j$ commute if $i \neq j$ for any $x, y \in H$.

(1) Consider $L \leq G$ generated by $\Gamma(H^n), A_n$ and $\{h_i h_j^{-1} \colon h \in H, i \neq j\}$. One easily checks that $L \lhd G$ and that $G/L \cong H_{\text{ab}} \times (\Sigma_n)_{\text{ab}}$, whence $\Gamma(G) \subseteq L$. Clearly $A_n, \Gamma(H^n) \leq \Gamma(G)$ and also $h_i h_j^{-1} = [h_i(ij), h_i^{-1}] \in \Gamma(G)$ so $L = \Gamma(G)$.

(2) We will show that the generators of $\Gamma(G)$ described in part (1) belong to $\Gamma(G^{\text{ev}})$. First, $\Gamma(H^n) \subseteq \Gamma(G^{\text{ev}})$ by Lemma A.4. For the other generators, we need to distinguish two cases. First assume that $k$ is even. Then $\Sigma_n \subseteq G^{\text{ev}}$ and therefore $A_n = \Gamma(\Sigma_n) \subseteq \Gamma(G^{\text{ev}})$ is a triviality. If $h \in H^{\text{ev}}$ and $i \neq j$ then $h_i h_j^{-1} = [h_i(ij), h_i^{-1}] \in \Gamma(G^{\text{ev}})$. If $h \in H^{\text{odd}}$ then for any $i \neq j$ find $\ell \neq i, j$ (note $n \geq 3$) and observe that $h_i h_j^{-1} = [h_\ell h_i(ij), (ij)] \in \Gamma(G^{\text{ev}})$. Therefore $\Gamma(G^{\text{ev}})$ contains all the generators of $\Gamma(G)$.

If $k$ is odd then $\Sigma_n \leq G$ contains odd permutations $(ij)$. For any $(ij\ell) \in A_n$ choose $h \in H^{\text{odd}}$ and note that $(ij\ell) = [(\ell i)h_i, (\ell j)h_j]$ so $A_n \subseteq \Gamma(G^{\text{ev}})$. For any $i \neq j$ choose $\ell \neq i, j$. If $h \in H^{\text{ev}}$ then $h_i h_j^{-1} = [h_i(\ell i j), h_i^{-1}] \in \Gamma(G^{\text{ev}})$. If $h \in H^{\text{odd}}$ then $h_i h_j^{-1} = [h_\ell(ij), h_j(ij)] \in \Gamma(G^{\text{ev}})$ and we deduce that $\Gamma(G^{\text{ev}})$ contains all the elements $h_i h_j^{-1}$ for all $h \in H$.

(3) Let $L$ be the subgroup of $G^{\text{ev}}$ generated by $\Gamma(H^2)$ and by $\{h_1 h_2^{-1} \colon h \in H^{\text{ev}}\}$. Clearly $L \lhd G$ and it is contained in $G^{\text{ev}}$. Now we fix once and for all an element $w \in H^{\text{odd}}$ if $k$ is odd and $w \in H^{\text{ev}}$ if $k$ is even. Let $\tau = (12)$ denote the non-identity element of $\Sigma_2$. Note that $w_1\tau \in G^{\text{ev}}$ and that $G^{\text{ev}}$ is generated by $(H^2)^{\text{ev}}$ and by $w_1\tau$. The image of $(H^2)^{\text{ev}}$ in $G^{\text{ev}}/L$ is clearly abelian and for any $u, v \in H$ such that $uv \in H^{\text{ev}}$ one checks that $[w_1\tau, u_1 v_2] = [w, v]_1 \cdot (vu^{-1})_1 (vu^{-1})_2^{-1} \in L$. We deduce that $G/L$ is abelian, so $L \supseteq \Gamma(G^{\text{ev}})$. By Lemma A.4, $\Gamma(H^2) = \Gamma((H^2)^{\text{ev}}) \subseteq \Gamma(G^{\text{ev}})$. Now, $h_1 h_2^{-1} = [h_1, \tau w_2] \in \Gamma(G^{\text{ev}})$ for any $h \in H^{ev}$ (note that $w$ was chosen according to the parity of $k$) whence $L = \Gamma(G^{\text{ev}})$. Since $\Gamma(G) \leq G^{\text{ev}}$ we obtain the exact sequence $0 \to \Gamma(G)/L \to {G^{\text{ev}}}_{\text{ab}} \to G_{\text{ab}} \xrightarrow{\text{sign}} C_2 \to 0$. By point (1), $\Gamma(G) = \langle L, y_1 y_2^{-1} \rangle$

for any $y \in H^{\mathrm{odd}}$ chosen arbitrarily. Also, $y_1 y_2^{-1} \notin L$ because $L \subseteq (H^{ev})^2$ and $(y_1 y_2^{-1})^2 \in L$ so $\Gamma(G)/L = C_2$. $\square$

In the next two results, $k$ denotes an algebraically closed field of characteristic $p$. Thus, $k^\times$ is a divisible group with $q$-torsion for all primes $q \neq p$.

**Lemma A.6.** *Fix $K \leq \Sigma_n$ and $p > 2$ and assume that one of the following holds.*

*(a) $K$ contains an odd permutation and $O^{p'}(K) \supseteq \Gamma^2(K)$.*

*(b) $K = K_1 \times K_2$ where both $K_1$ and $K_2$ contain odd permutations.*

*(c) $K = H \wr \Sigma_r$ where $r \geq 3$ and $H \leq \Sigma_m$ contains odd permutations.*

*Then the inclusion $K^{\mathrm{ev}} \to K$ induces a short exact sequence*

$$0 \to C_2 \to \mathrm{Hom}(K, k^\times) \to \mathrm{Hom}(K^{\mathrm{ev}}, k^\times) \to 0.$$

*The kernel $C_2$ is generated by the signature map $K \to C_2 \leq k^\times$.*

*Proof.* (a) It is clear that $O^{p'}(K^{\mathrm{ev}}) = O^{p'}(K)$ because $p > 2 = |K \colon K^{\mathrm{ev}}|$. We obtain a short exact sequence of abelian groups $0 \to \frac{K^{\mathrm{ev}}}{O^{p'}(K)} \to \frac{K}{O^{p'}(K)} \to C_2 \to 0$. The results follows by applying the exact functor $\mathrm{Hom}(-, k^\times)$. Similarly, (b) and (c) follow from Lemma A.4 and Proposition A.5(2) by applying the exact functor $\mathrm{Hom}(-, k^\times)$. Note that $k^\times$ contains 2-torsion. $\square$

**Lemma A.7.** *Fix $p > 2$ and $H \leq \Sigma_n$ such that $H^{\mathrm{odd}} \neq \emptyset$ and set $G = H \wr \Sigma_2$. Then the inclusion $i \colon G^{\mathrm{ev}} \to G$ induces a short exact sequence of abelian groups*

$$0 \to C_2 \to \mathrm{Hom}(G, k^\times) \xrightarrow{i^*} \mathrm{Hom}(G^{\mathrm{ev}}, k^\times) \to C_2 \to 0.$$

*The first map is induced by $G \xrightarrow{\mathrm{sign}} C_2$. The last map is obtained by applying* $\mathrm{Hom}(-, k^\times)$ *to the map $H/H^{\mathrm{ev}} \to (G^{\mathrm{ev}})_{\mathrm{ab}}$ induced by $H \xrightarrow{h \mapsto (h, h^{-1})} G^{\mathrm{ev}}$.*

*Proof.* Apply the exact functor $\mathrm{Hom}(-, k^\times)$ to the exact sequence in Proposition A.5(3). $\square$

**Proposition A.8.** *Fix some $K \leq \Sigma_e$ and a finite group $H$ and consider a subgroup $G$ of $\Gamma := H \wr K$. Assume that the projection $\Gamma \to K$ carries $G$* isomorphically *onto $E \leq K$ where $E$ is an elementary abelian $p$-subgroup of $\Sigma_e$ which acts freely and transitively. Then there exists some $\mathbf{h}$ in the base group $H^e \leq \Gamma$ which conjugates $G$ to a subgroup of $K \leq \Gamma$.*

*Proof.* Assume that $E \cong (\mathbb{Z}/p)^d$. We may identify $\Sigma_e = \mathrm{Sym}(E)$ and identify $E$ with the image of $E \to \mathrm{Sym}(E)$ via left translations. We view the elements of the base group $H^e$ as set-functions $\mathbf{h} \colon E \to H$ (not homomorphisms!). Product of elements is taken pointwise and $\Sigma_e$ acts by permuting the factors, namely $\sigma(\mathbf{h})(x) = \mathbf{h}(\sigma x)$ for any $x \in E$ and $\sigma \in \Sigma_e$. If $\sigma \in E$ then $\sigma x$ is indeed the product $\sigma \cdot x$ in $E$. Let $\mathbf{1}$ denote the identity of $H^e$.

Let $g_1, \ldots, g_d$ be generators for $G$ and let $\sigma_, \ldots, \sigma_d \in E$ be their images in $E$. Note that $E \leq K$ is a subgroup of $\Gamma = H \wr K$ in a natural way. Thus, there are $\mathbf{h}_1, \ldots, \mathbf{h}_d \in H^e$ such that $g_i = \mathbf{h}_i \sigma_i$ for all $i$. Since $g_i^p = 1$ for all $i$ and $[g_i, g_j] = 1$ for all $i, j$ we obtain the following relations

$$\mathbf{h}_i \cdot \sigma_i(\mathbf{h}_i) \cdot \sigma_i^2(\mathbf{h}_i) \cdot \dots \cdot \sigma_i^{p-1}(\mathbf{h}_i) = \mathbf{1} \tag{A.8.1}$$

$$\mathbf{h}_i \cdot \sigma_i(\mathbf{h}_j) = \mathbf{h}_j \cdot \sigma_j(\mathbf{h}_i). \tag{A.8.2}$$

Our goal is to prove that by possibly replacing $g_1, \ldots, g_d$ with $H^e$-conjugates, we have $g_i = \sigma_i$ for all $i$, or in other words $\mathbf{h}_i = \mathbf{1}$.

Assume that this is not the case and assume that $g_1, \ldots, g_d$ have the property that $\mathbf{h}_i = \mathbf{1}$, namely $g_i = \sigma_i$, for all $i < k$ where $k$ is maximal with this property. By assumption $k \leq d$ and $\mathbf{h}_k \neq \mathbf{1}$. Let $E_i$ be the subgroup of $E$ generated by $\sigma_i$ and set $F = E_1 \oplus \cdots \oplus E_{k-1}$. Then $Y := E_{k+1} \oplus \cdots \oplus E_d$ is a set of representatives for the cosets of $F \oplus E_k$ in $E$. Define an element $\mathbf{r} \in H^e$ as follows. Every $x \in E$ has the form $f\sigma_k^s y$ for unique $f \in F, y \in Y$ and $0 \leq s \leq p-1$. Define

$$\mathbf{r}(x) = \mathbf{h}_k(fy) \cdot \mathbf{h}_k(f\sigma_k y) \cdot \mathbf{h}_k(f\sigma_k^2 y) \cdot \dots \cdot \mathbf{h}_k(f\sigma_k^{s-1} y).$$

Note that $\mathbf{r}(fy) = 1$. Fix some $i < k$. Since $\mathbf{h}_i = \mathbf{1}$, relation (A.8.2) implies that $\sigma_i(\mathbf{h}_k) = \mathbf{h}_k$, i.e. $\mathbf{h}_k(x) = \mathbf{h}_k(\sigma_i x)$. Since $\sigma_i \in F$, for any $x \in E$ of the form $f\sigma_k^s y$,

$$(\sigma_i(\mathbf{r}))(x) = \mathbf{r}(\sigma_i f \sigma_k^s y) = \prod_{j=0}^{s-1} \mathbf{h}_k(\sigma_i f \sigma_k^j y) = \prod_{j=0}^{s-1} \mathbf{h}_k(f\sigma_k^j y) = \mathbf{r}(x).$$

We deduce that if $i < k$ then $\mathbf{r} \cdot g_i \cdot \mathbf{r}^{-1} = \mathbf{r} \cdot \sigma_i \cdot \mathbf{r}^{-1} = \mathbf{r} \cdot \sigma_i(\mathbf{r}^{-1}) \cdot \sigma_i = \sigma_i$. Similarly, for $x = f\sigma_k^s y$

$$(\mathbf{r} \cdot \mathbf{h}_k)(x) = (\prod_{j=0}^{s} \mathbf{h}_k(f\sigma_k^j y)).$$

If $0 \leq s \leq p-2$ then the expression on the right is precisely the expression for $\mathbf{r}(\sigma_k x)$. If $s = p-1$ then by (A.8.1) this expression is equal to $1 \in H$ which in turn, is $\mathbf{r}(\sigma_k x)$ because $\sigma_k x = f\sigma_k^p y = fy$. Thus $\mathbf{r} \cdot \mathbf{h}_k = \sigma_k(\mathbf{r})$. We deduce that $\mathbf{r} \cdot g_k \cdot \mathbf{r}^{-1} = \mathbf{r} \cdot \mathbf{h}_k \cdot \sigma_k(\mathbf{r}^{-1}) \cdot \sigma_k = \mathbf{1} \cdot \sigma_k = \sigma_k$. We have thus shown that $\mathbf{r} \cdot g_i \cdot \mathbf{r}^{-1} = \sigma_i$ for all $i \leq k$ which contradicts the maximality of $k$. $\square$

**Lemma A.9.** *Set $q = p^r$ for a prime $p$ and $G = \mathrm{GL}_n(q)$. Then $O^{p'}(G) = \mathrm{SL}_n(q)$.*

*Proof.* The elements of $\mathrm{GL}_n(\mathbb{F}_q)$ of $p$-power order are precisely the transvections which, in turn, generate $SL_n(\mathbb{F}_q)$. $\square$

## B. Proof of Proposition 2.12

Consider $N \lhd G$ and set $\bar{G} = G/N$. Let $\bar{K}$ denote the image of $K \leq G$ in $\bar{G}$. For any $G$-set $X$ set $\bar{X} := X/N$. Clearly $\bar{X}$ is a $\bar{G}$-set, hence a $G$-set. The assignment $X \mapsto \bar{X}$ is a functor $q\colon \{G\text{-sets}\} \to \{\bar{G}\text{-sets}\}$. The projection $\mathbb{Z}[X] \to \mathbb{Z}[\bar{X}]$ is a homomorphism of $G$-modules.

**Proposition B.1.** *Fix $N \lhd G$ and set $\bar{G} = G/N$. Let $\bar{K}$ be the image of $K \leq G$ in $\bar{G}$. Fix an abelian group $A$ and set $\mathcal{H}^1_G := \mathcal{H}^1_G(A), \mathcal{H}^1_N := \mathcal{H}^1_N(A)$ etc. Then*

1. *If $X = G/K$ then there is an isomorphism $\mathcal{H}^1_G(X) \cong \mathrm{Hom}(K, A)$ natural in $K$. Also, $\mathrm{fix}_{\bar{G}}(\mathcal{H}^1_{G|N})(X) \cong \mathrm{Hom}(K \cap N, A)^{\bar{K}}$ where $\bar{K}$ acts on $K \cap N$ by conjugation. The map $\mathcal{H}^1_G(X) \to \mathrm{fix}_{\bar{G}}(\mathcal{H}^1_{G|N})(X)$ defined in 2.11, is induced by applying $\mathrm{Hom}(-, A)$ to the inclusion $N \cap K \to K$.*
2. *There is a natural transformation $\mathcal{H}^i_{\bar{G}} \circ q \to \mathcal{H}^i_G$ of coefficient functors for $G$. If $X = G/K$ then $\mathcal{H}^1_{\bar{G}} \circ q(X) \cong \mathrm{Hom}(\bar{K}, A)$ and the map to $\mathcal{H}^1_G(X)$ is obtained by applying $\mathrm{Hom}(-, A)$ to the projection $K \to \bar{K}$.*

*Proof.* (1) Shapiro's lemma implies that $\mathrm{Ext}^1_G(\mathbb{Z}[G/K], A) = \mathrm{Ext}^1_K(\mathbb{Z}, A)$ because $\mathbb{Z}[G/K] = \mathbb{Z}\uparrow^G_K$. Now, $H^1(K, A) = \mathrm{Hom}(K, A)$ because $A$ is a trivial $K$-module.

Clearly, $X\downarrow^G_N = \coprod_{g \in G/KN} N/N \cap {}^gK$ where ${}^gK = gKg^{-1}$ and $\bar{G}$ acts transitively on the $N$-orbits via left translation. Also, $\bar{K} \leq \bar{G}$ is the isotropy group of the orbit $N/N \cap K$ in $X\downarrow^G_N$. Therefore

$$\mathcal{H}^1_{G|N}(X) = \Big( \oplus_{g \in G/KN}\, \mathcal{H}^1_N(\mathbb{Z}[N/N \cap {}^gK]) \Big)^{\bar{G}} \cong \mathrm{Hom}(N \cap K, A)^{\bar{K}}$$

(2) Let $\mathbb{Z}[G]_\bullet$ denote the bar construction. It is a projective resolution of the trivial $G$-module $\mathbb{Z}$ which consists of $G{-}G$ bimodules and it is a split exact chain complex of abelian groups. For any $G$-module $M$ consider the cochain complex $E(M) := \mathrm{Hom}_G(M, \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}[G]_\bullet, A))$. Now, $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}[G]_\bullet, A)$ is a split exact cochain complex and therefore $H^0E(M) = \mathrm{Hom}(M, A)$ because $\mathrm{Hom}_G(M, -)$ is left exact. If $M$ is projective then $H^{*\geq 1}E(M) = 0$ because $\mathrm{Hom}_G(M, -)$ is exact. We deduce that $H^*E(M) \cong \mathrm{Ext}^*_G(M, A)$.

The epimorphism $G \to \bar{G}$ gives rise to a a morphism $\mathbb{Z}[G]_\bullet \to \mathbb{Z}[\bar{G}]_\bullet$ of cochain complexes of $G - G$-bimodules. Therefore, for any $G$-set $X$ we obtain a morphism of cochain complexes

$$\mathrm{Hom}_G(\mathbb{Z}[\bar{X}], \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}[\bar{G}]_\bullet, A)) \to \mathrm{Hom}_G(\mathbb{Z}[X], \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}[G]_\bullet, A)).$$

Since both $\mathbb{Z}[\bar{X}]$ and $\mathbb{Z}[\bar{G}]_\bullet$ have a trivial action of $N$, the left hand side is equal to $\mathrm{Hom}_{\bar{G}}(\mathbb{Z}[\bar{X}], \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}[\bar{G}]_\bullet, A))$ and by taking homology groups we obtain a natural transformation $\mathrm{Ext}^*_{\bar{G}}(\mathbb{Z}[\bar{X}], A) \to \mathrm{Ext}^*_G(\mathbb{Z}[X], A)$. In other words, we obtain a natural transformation of coefficient functors $\mathcal{H}^i_{\bar{G}}(A) \to \mathcal{H}^i_G(A)$. Inspection of the isomorphism in (1) shows that $\mathcal{H}^1_{\bar{G}}(A)(\bar{X}) \to \mathcal{H}^1_G(A)(X)$ is $\mathrm{Hom}(\bar{K}, A) \to \mathrm{Hom}(K, A)$ induced by $K \to \bar{K}$. $\square$

***Proof of Proposition 2.12.*** Recall that $G = N^r \rtimes \Sigma_r$. By Definition 2.11 and Proposition B.1(2) we obtain natural transformations $\mathcal{H}^1_{\Sigma_r} \circ q \to \mathcal{H}^1_G \to \mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{N^r})$. Our goal is to show that when these natural maps are evaluated on an orbit of a simplex $x$ in $X$, they give rise to a short exact sequence of abelian groups.

By possibly replacing $x$ with another simplex in its $G$-orbit we may assume that

$$\mathbf{x} = (\underbrace{y^1, \dots, y^1}_{n_1 \text{ times}}, \dots, \underbrace{y^s, \dots, y^s}_{n_s \text{ times}})$$

where $y^1, \dots, y^s$ are simplices in $Y$ in *distinct* orbits of $N$. Set $H_i = \mathrm{Iso}_N(y^i)$ and observe that $K := \mathrm{Iso}_G(x) = \prod_{i=1}^s H_i \wr \Sigma_{n_i}$. Therefore $K \cap N^r = \prod_{i=1}^s H_i^{n_i}$ and $\bar{K} = \prod_{i=1}^s \Sigma_{n_i}$ acts on $K \cap N^r$ by permuting the factors. It now follows from Propositions A.5(1) and B.1(1) that $0 \to \mathcal{H}^1_{\Sigma_r} \circ q(X) \to \mathcal{H}^1_G(X) \to \mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{G|N^r}(X)) \to 0$ is a split short exact sequence. $\square$

## C. Calculation of some Bredon cohomology groups

Fix a prime $p$ and $d \geq 2$. Set $K = \Sigma_{p^d}$ and recall the definition of the $K$-poset $\mathcal{B}(d)$ and the $K \wr \Sigma_r$-posets $\mathcal{B}(d)^r$ and $\mathcal{B}(d)^r_0$ from 4.2. The goal of this section is the following result.

**Proposition C.1.** *Fix $d \geq 2$ and $r \geq 1$ and set $K = \Sigma_{p^d}$ and $G = K \wr \Sigma_r$. Then with the notation of Definition 2.11, where $k$ is an algebraically closed field of characteristic $p$,*

$$H^{*\geq 1}_{K\wr\Sigma_r}(|\mathcal{B}(d)^r|; \mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{G|K^r}(k^\times))\,) = H^{*\geq 1}_{K\wr\Sigma_r}(|\mathcal{B}(d)^r_0|; \mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{G|K^r}(k^\times))\,) = 0.$$

Let us first outline the calculation. Details will follow later.

**Construction C.2.** Let $F \colon \mathcal{C} \to \mathbf{Ab}$ be a functor. For a subcategory $\mathcal{D} \subseteq \mathcal{C}$ let $C^*(\mathcal{D}; F)$ denote the cobar construction of $F|_{\mathcal{D}}$ which is described in [3, XI.5.1 and 6.2] or in [11, Apendix II.3.3] for its dual. Thus, $C^n(\mathcal{D}; F) = \prod_{d_\bullet} F(d_n)$ where the product runs through all the $n$-simplices $d_\bullet$ in $|\mathcal{D}|$ of the form $d_0 \to \cdots \to d_n$. More generally, if $\mathcal{D}' \subseteq \mathcal{D}$ then $C^*(\mathcal{D}, \mathcal{D}'; F)$ is the kernel of the surjective map $C^*(\mathcal{D}; F) \to C^*(D'; F)$. There results the usual long exact sequence in homology and $C^n(\mathcal{D}, \mathcal{D}'; F) = \prod_{d_\bullet} F(d_n)$ where $d_\bullet \in |\mathcal{D}|_n - |\mathcal{D}'|_n$.

Let $\mathcal{C}^r$ denote the $r$-fold product of $\mathcal{C}$ and define a functor $F^{\otimes r} \colon \mathcal{C}^r \to \mathbf{Ab}$

$$F^{\otimes r}(c^1, \dots, c^r) = \prod_{i=1}^r F(c^i).$$

Thus, if $\pi_i \colon \mathcal{C}^r \to \mathcal{C}$ denotes the projection to the $i^{\text{th}}$ factor then $F^{\otimes r} = \oplus_{i=1}^r \pi_i^*(F)$. Clearly $\Sigma_r$ acts on $\mathcal{C}^r$ and on the set $\{\pi_1, \dots, \pi_r\}$ in such a way that $\pi_i^*(F)(\sigma(\mathbf{c})) = \pi^*_{\sigma^{-1}(i)}(F)(\mathbf{c})$ for any $\mathbf{c} \in \mathcal{C}$ and any $\sigma \in \Sigma_r$. In this way, for any subcategory $\mathcal{D} \subseteq \mathcal{C}^r$ invariant under $\Sigma_r$, the cobar construction $C^*(\mathcal{D}; F^{\otimes r})$ becomes in a natural way a cochain complex of $\Sigma_r$-modules.

**Proposition C.3.** *With the notation of Construction C.2, the assignments $F \mapsto C^*(\mathcal{D}, D'; F^{\otimes r})$ and $F \mapsto C^*(\mathcal{D}, D'; F^{\otimes r})^{\Sigma_r}$ are exact functors.*

In this section we will study the vanishing of the homology of the cochain complex of the invariants $C^*(\mathcal{D}; F^{\otimes n})^{\Sigma_n}$ in some special cases.

Recall the definition of the poset $\mathcal{P}art(d)$ from 4.3. If $\mathbf{c} \leq \mathbf{c}'$ in $\mathcal{P}art(d)$ then there is a surjective function of sets $\mathbf{c}' \to \mathbf{c}$ which maps an interval $i' \in \mathbf{c}'$ to the unique interval $i \in \mathbf{c}$ which contains it. If $L$ is an abelian group and $X$ is a set, let $L^X$ denote the group of all the functions $X \to L$.

**Definition C.4.** Fix a finite abelian group $L$. Let $\kappa_L \colon \mathcal{P}art(d) \to \mathbf{Ab}$ be the functor $\mathbf{c} \mapsto L^{\mathbf{c}}$. Thus, $\kappa_L(\mathbf{c}) \cong L^n$ if $\mathbf{c} = (c_1, \dots, c_n)$.

The key to the proof of Proposition C.1 is the next result.

**Proposition C.5.** *Fix $d \geq 2$ and let $\mathcal{D}$ denote either one of the $\Sigma_r$-posets $\mathcal{P}art(d)^r$ or $\mathcal{P}art(d)_0^r$ (Definition 4.3). Then the cohomology groups of the cochain complex $C^*(\mathcal{D}; \kappa_L^{\otimes r})^{\Sigma_r}$ vanish for all $* \geq 1$.*

In any $G$-poset $\mathcal{C}$ we will denote the orbits of its $n$-simplices, namely the simplices of $|\mathcal{C}|/G$ by $[c_0 \leq \dots \leq c_n]$.

**Proposition and Definition C.6.** *Let $\mathcal{C}$ be a $G$-poset and let $\mathcal{M}$ be a coefficient functor. Let $\bar{\mathcal{C}}$ denote the quotient poset $\mathcal{C}/G$. Assume that*

*(a) $|\mathcal{C}|/G \to |\bar{\mathcal{C}}|$ is an isomorphism of simplicial sets.*

*(b) Applying $\mathcal{M}$ to the $G$-maps $[c_0 \leq \dots \leq c_n] \to [c_n]$ yields isomorphisms of abelian groups.*

*Then the assignment $[c] \mapsto \mathcal{M}([c])$ gives rise to a functor $\Phi_{\mathcal{M},\mathcal{C}} \colon \bar{\mathcal{C}} \to \mathbf{Ab}$.*

Fix a finite group $G$ and recall from Definition 2.1 the coefficient functor $\mathcal{H}^1_G(k^\times)$. For $r \geq 1$ we obtain a coefficient functor $\mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{G\wr\Sigma_r|G^r}(k^\times))$.

**Lemma C.7.** *Let $\mathcal{C}$ be a $G$-poset and assume that the hypotheses of Definition C.6 hold for $\mathcal{C}$ and $\mathcal{H}_G := \mathcal{H}^1_G(k^\times)$ and set $\bar{\mathcal{C}} = \mathcal{C}/G$. Set $\Gamma = G \wr \Sigma_r$ and let $\bar{\mathcal{D}} \subseteq \bar{\mathcal{C}}^r$ be a $\Sigma_r$-subposet and let $\mathcal{D}$ be its preimage in the $\Gamma$-poset $\mathcal{C}^r$. Then*

$$C^*_{G\wr\Sigma_r}(|\mathcal{D}|; \mathrm{fix}_{\Sigma_r}(\mathcal{H}_{\Gamma|G^r})) \cong C^*(\bar{\mathcal{D}}; (\Phi_{\mathcal{H}_G,\mathcal{C}})^{\otimes r})^{\Sigma_r}$$

Assuming C.3–C.7 we can prove the main result of this section.

***Proof of Proposition C.1.*** Proposition 4.5 implies that $\mathcal{B}(d)/K = \mathcal{P}art(d)$ and $|\mathcal{B}(d)|/K = |\mathcal{P}art(d)|$. Also, if $P_0 \leq \dots \leq P_s$ is an $s$-simplex in $\mathcal{B}(d)$, then it is in the same orbit of $V(\mathbf{c}_0) \leq \dots \leq V(\mathbf{c}_s)$ and there is some $Q \leq V(\mathbf{c}_s)$ such that

$$N_K(V(\mathbf{c}_0) \leq \dots \leq V(\mathbf{c}_s)) = Q \rtimes \mathrm{GL}_{\mathbf{c}_s}(p) \quad \leq \quad V(\mathbf{c}_s) \rtimes \mathrm{GL}_{\mathbf{c}_s}(p) = N_K(V(\mathbf{c}_s)).$$

Set $\mathcal{H}_K := \mathcal{H}^1_K(k^\times)$. Since $k^\times$ contains no $p$-torsion we deduce that

$$\mathcal{H}_K([P_0 \le \cdots \le P_s]) = \mathrm{Hom}(Q \rtimes \mathrm{GL}_{\mathbf{c}_s}(p), k^\times) = \mathrm{Hom}(\mathrm{GL}_{\mathbf{c}_s}(p), k^\times) = \\ \mathrm{Hom}(V(\mathbf{c}_s) \rtimes \mathrm{GL}_{\mathbf{c}_s}(p), k^\times) = \mathcal{H}_K([P_s]).$$

Set $\Gamma = K \wr \Sigma_r$. Let $\mathcal{D}$ denote either $\mathcal{B}(d)^r$ or $\mathcal{B}(d)^r_0$. Denote $\bar{\mathcal{D}} = \mathcal{D}/K^r$, thus either $\bar{\mathcal{D}} = \mathcal{P}art(d)^r$ or $\bar{\mathcal{D}} = \mathcal{P}art(d)^r_0$. We now apply Lemma C.7 to deduce that

$$C^*_\Gamma(|\mathcal{D}|; \mathrm{fix}_{\Sigma_r}(\mathcal{H}_{\Gamma|K^r})) = C^*(\bar{\mathcal{D}}, (\Phi_{\mathcal{H}_K,\mathcal{B}(d)})^{\otimes r})^{\Sigma_r}.$$

The functor $\Phi_{\mathcal{H}_K,\mathcal{B}(d)}$ is defined in Proposition C.6. By Proposition 4.5 and Lemma A.9 it has the following form, where $\mathbf{c} \in \mathcal{P}art(d) = \mathcal{B}(d)/K$,

$$\Phi_{\mathcal{H}_K,\mathcal{B}(d)} \colon \mathbf{c} \mapsto \mathrm{Hom}(V(\mathbf{c}) \rtimes \mathrm{GL}_{\mathbf{c}}(p), k^\times) = \mathrm{Hom}(\prod_i \mathrm{GL}_{c_i}(p), k^\times) = (\mathbb{F}_p^\times)^{\mathbf{c}}.$$

We deduce, see Definition C.4, that $\Phi_{\mathcal{H}_K,\mathcal{B}(d)} = \kappa_L$ where $L = \mathbb{F}_p^\times$. Now the result follows from Proposition C.5. □

Let us now fill in the details for C.3, C.5, C.6 and C.7.

***Proof of Proposition C.6.*** For any $[c_1] \le [c_2]$ in $\bar{\mathcal{C}}$, hypothesis (a) implies that there is a unique orbit of a 1-simplex $[c'_1 \le c'_2]$ such that $[c'_1] = [c_1]$ and $[c'_2] = [c_2]$. We can therefore unambiguously define $\Phi_{\mathcal{M},\mathcal{C}}$ on the morphism $[c_1] \le [c_2]$ in $\bar{\mathcal{C}}$ by

$$\mathcal{M}([c_1]) = \mathcal{M}([c'_1]) \to \mathcal{M}([c'_1 \le c'_2]) \cong \mathcal{M}([c'_2]) = \mathcal{M}([c_2]).$$

Observe that if $I_0 = \{c_1 \le \cdots \le c_n\}$ is a linearly ordered subset of $\mathcal{C}$ whose orbit in $\mathcal{C}^n$ is $[I_0]$, then for any $I_1, I_2 \subseteq I_0$ the $G$-maps $[I_0] \to [I_1] \to [I_1 \cap I_2]$ and $[I_0] \to [I_2] \to [I_1 \cap I_2]$ are equal. We leave it for the reader to deduce that $\Phi_{\mathcal{M},\mathcal{C}}$ respects compositions. □

**Observation C.8.** Consider a poset $\mathcal{C}$ and a functor $F \colon \mathcal{C} \to \mathbf{Ab}$. It follows directly from the definition of $F^{\otimes r}$, see Construction C.2, that $C^*(\mathcal{C}^r; F^{\otimes r})$ is isomorphic to $\prod_r C^*(\mathcal{C}; F)$ with $\Sigma_r$ acting by permuting the factors.

Consider a $\Sigma_r$-invariant subposet $\mathcal{D} \subseteq \mathcal{C}^r$. Observe that $C^n(\mathcal{D}; F^{\otimes r})$ is a direct product of $\Sigma_r$-modules indexed by the orbits $[d_\bullet]$ of the $n$-simplices of $|\mathcal{D}|$. Since $\Sigma_r$ acts by permuting the factors of $\mathcal{C}^r$, the isotropy group of an $n$-simplex $\mathbf{d}_0 \le \cdots \le \mathbf{d}_n$ has the form $\Sigma_{k_1} \times \cdots \times \Sigma_{k_s}$ where $k_1 + \cdots + k_s = r$ and

$$d_n = (\underbrace{e^1, \dots, e^1}_{k_1 \text{ times}}, \dots, \underbrace{e^s, \dots, e^s}_{k_s \text{ times}}).$$

Now, $F^{\otimes r}(\mathbf{d}_n) = \prod_{k_1} F(e^1) \times \cdots \times \prod_{k_s} F(e^s)$ on which $\mathrm{Iso}_{\Sigma_r}(\mathbf{d}_\bullet) = \Sigma_{e_1} \times \cdots \times \Sigma_{k_s}$ acts by permuting the factors. Since $\Sigma_r$ acts transitively on $[\mathbf{d}_\bullet]$,

$$C^n(\mathcal{D}; F^{\otimes r})^{\Sigma_r} \cong \prod_{[\mathbf{d}_\bullet] \subseteq |\mathcal{D}|_n} F(\mathbf{d}_n)^{\mathrm{Iso}_{\Sigma_r}(\mathbf{d}_\bullet)}$$

and with the notation above, $F(\mathbf{d}_n)^{\mathrm{Iso}_{\Sigma_r}(\mathbf{d}_\bullet)} \cong \prod_{i=1}^s F(e^i)$. From these formulas one easily checks that if $\mathcal{D}$ is a $\Sigma_r$-subposet of $\mathcal{C}^r$ and $\mathrm{Const}_A\colon \mathcal{C} \to \mathbf{Ab}$ is the constant functor with value $A$ then

$$C^*(\mathcal{D}; \mathrm{Const}_A^{\otimes r})^{\Sigma_r} \cong C^*_{\Sigma_r}(|\mathcal{D}|; \mathcal{H}^0(M))$$

where $M$ is the $\Sigma_r$-permutation module $\oplus_r A$, see Definition 2.1.

***Proof of Proposition C.3.*** This is immediate from Observation C.8 because for any $n$-simplex $\mathbf{d}_\bullet$, the assignment $F \mapsto \prod_{i=1}^s F(e_s)$ is an exact functor. $\square$

***Proof of Lemma C.7.*** With the notation of Observation C.8, for every $n$-simplex $\mathbf{d}_\bullet$ in $|\mathcal{D}|$ we deduce from Proposition 2.10 that

$$\begin{aligned}\mathrm{fix}_{\Sigma_r}(\mathcal{H}^1_{\Gamma|G^r}(k^\times))([d_\bullet]) = \mathcal{H}^1_{G^r}([d^1_\bullet]^{k_1} \times \cdots \times [d^s_\bullet]^{k_s})^{\Sigma_{k_1}\times\cdots\times\Sigma_{k_s}} = \\ (\mathcal{H}_G([d^1_\bullet])^{k_1} \times \cdots \times \mathcal{H}_G([d^s_\bullet])^{k_s})^{\Sigma_{k_1}\times\cdots\times\Sigma_{k_s}} = (\Phi_{\mathcal{H}_G,\mathcal{C}})^{\otimes r}([d_\bullet])^{\Sigma_r},\end{aligned}$$

where $\mathbf{d}_\bullet$ has the form $\prod_{k_1} \mathbf{d}^1_\bullet \times \cdots \times \prod_{k_s} \mathbf{d}^s_\bullet$ for distinct $n$-simplices $d^i_\bullet$ in $|\mathcal{C}|$. By inspection of the definition of the cochain complex $C^*_{\Sigma_r}(|\mathcal{D}|; \mathrm{fix}_{\Sigma_r}(\mathcal{H}_{\Gamma|G^r}))$ and the description of $C^*(\bar{\mathcal{D}}; (\Phi_{\mathcal{H}_G,\mathcal{C}})^{\otimes r})^{\Sigma_r}$ in Observation C.8, one sees that these cochain complexes are isomorphic. $\square$

Recall that a (co)chain complex is called acyclic if all its homology groups vanish.

***Proof of Proposition C.5.*** Let $\mathcal{T}(d)$ be the poset whose objects are pairs $(\mathbf{c}, I)$ where $\mathbf{c} \in \mathcal{P}art(d)$ and $I \in \mathbf{c}$ is an interval, and $(\mathbf{c}, I) \preceq (\mathbf{d}, J)$ if $\mathbf{c} \preceq \mathbf{d}$, namely $\mathbf{d}$ refines $\mathbf{c}$ and $J \subseteq I$. By inspection

$$C^*(\mathcal{P}art(d); \kappa_L) = C^*(\mathcal{T}(d); \mathrm{Const}_L).$$

We will write $[i, j]$ for the interval $\{i, i+1, \dots, j\}$ in $\{1, \dots, d\}$. Clearly the object $((d), [1, d])$ is the minimum of $\mathcal{T}(d)$. In particular $\mathcal{T}(d)|$ is contractible. Let $\mathcal{T}_0(d)$ denote the sub-poset of $\mathcal{T}(d)$ obtained by removing the minimum element. For any $r \geq 1$ the poset $\mathcal{T}(d)^r$ denotes the $r$-fold product of $\mathcal{T}(d)$ and $\mathcal{T}(d)^r_0$ is obtained by removing the minimum of $\mathcal{T}(d)^r$. By Observation C.8 we now obtain

$$(\dagger) \qquad C^*(\mathcal{P}art(d)^r; \kappa_L^{\otimes r})^{\Sigma_r} = C^*(\mathcal{T}(d)^r; \mathrm{Const}_L^{\otimes r})^{\Sigma_r} = C^*_{\Sigma_r}(|\mathcal{T}(d)^r|; \mathcal{H}^0(M)),$$

$$(\ddagger) \qquad C^*(\mathcal{P}art(d)^r_0; \kappa_L^{\otimes r})^{\Sigma_r} = C^*(\mathcal{T}(d)^r_0; \mathrm{Const}_L^{\otimes r})^{\Sigma_r} = C^*_{\Sigma_r}(|\mathcal{T}(d)^r_0|; \mathcal{H}^0(M))$$

where $M$ is the $\Sigma_r$-module $L^r$ via permutation of the factors.

Since the $\Sigma_r$-poset $\mathcal{T}(d)^r$ has a minimum, its nerve is $\Sigma_r$-equivariantly contractible and together with ($\dagger$) the Proposition follows in the case $\mathcal{D} = \mathcal{P}art(d)^r$. For the remainder of the proof we therefore consider the case $\mathcal{D} = \mathcal{P}art(d)^r_0$.
**Case I:** $d > 2$. We will prove that $|\mathcal{T}(d)^r_0|$ is $\Sigma_r$-equivalent to a point and therefore the result follows from ($\ddagger$).

Let $\mathcal{I}(d)$ denote the poset of the intervals in $[1,d]$ ordered by reverse inclusion, namely $I \preceq J$ if $J \subseteq I$. Let $\mathcal{I}_0(d)$ denote $\mathcal{I}(d)$ with the minimum element $[1,d]$ removed. Let $\mathcal{I}(d)^r$ denote the $r$-fold product of $\mathcal{I}(d)$ and let $\mathcal{I}(d)^r_0$ denote the poset obtained by removing from $\mathcal{I}(d)^r$ its minimum element $([1,d],\dots,[1,d])$.

The poset $\mathcal{I}_0(d)$ contains two minimal elements $\mu_1 = [1,d-1]$ and $\mu_2 = [2,d]$. We obtain two sub-posets $\mathcal{I}_0(d)_{\geq \mu_1}$ and $\mathcal{I}_0(d)_{\geq \mu_2}$ whose intersection is the poset $\mathcal{I}_0(d)_{\geq \mu_3}$ where $\mu_3 = [2,d-1]$ (here is where $d \geq 3$ is needed). It follows that $|\mathcal{I}_0(d)|$ is the union of $|\mathcal{I}_0(d)_{\geq \mu_1}|$ and $|\mathcal{I}_0(d)_{\geq \mu_2}|$ along $|\mathcal{I}_0(d)_{\geq \mu_3}|$, all three of them are contractible because these posets contain maxima. Thus, $|\mathcal{I}_0(d)|$ is the homotopy pushout of contractible spaces and it is therefore contractible.

Next consider the functor (morphism of posets) $\rho\colon \mathcal{T}_0(d) \to \mathcal{I}_0(d)$ defined by $(\mathbf{c}, I) \mapsto I$. Given $J \in \mathcal{I}_0(d)$ the over-category $(\rho \downarrow J)$, see [22, §II.6], is the subposet of $\mathcal{T}_0(d)$ consisting of the objects $(\mathbf{c}, I)$ such that $I \preceq J$, namely $J \subseteq I$. This poset contains a maximum $(\mathbf{d}, J)$ where $\mathbf{d}$ is the partition of $[1,d]$ into the interval $J$ and singletons. Its nerve is therefore contractible and by [25, Proposition 1.6] we obtain $|\mathcal{T}_0(d)| \simeq |\mathcal{I}_0(d)| \simeq *$.

Next we claim that $|\mathcal{T}(d)^s_0| \simeq *$ for any $s \geq 1$. Let $\mathcal{P}(s)_{<s}$ denote the poset of all the subsets (including the empty set) of $\{1,\dots,s\}$ of size $< s$ ordered by inclusion. Given $R \in \mathcal{P}(s)_{<s}$ let $\bar{R}$ denote its complement and let

$$F(R) = \prod_R |\mathcal{T}(d)| \times \prod_{\bar{R}} |\mathcal{T}_0(d)|$$

be the obvious subspace of $|\mathcal{T}(d)^r|$. If $R' \subseteq R$ then $F(R') \subseteq F(R)$ and we obtain a functor $F\colon \mathcal{P}(s)_{<s} \to \mathbf{Top}$. Note that $R \cap R'$ is the greatest lower bound of $R$ and $R'$ and that $F(R \cap R') = F(R) \cap F(R)$. Thus, $F$ is a "free diagram" in the language of [9], or alternatively $F$ is a Reedy cofibrant functor, see e.g. [10, §VI.22] so $\operatorname{colim} F \simeq \operatorname{hocolim} F$. We deduce that

$$|T(d)^s_0| = \bigcup_{R \in \mathcal{P}(s)_{<s}} F(R) = \operatorname*{colim}_{\mathcal{P}(s)_{<s}} F \simeq \operatorname*{hocolim}_{\mathcal{P}(s)_{<s}} F \simeq \operatorname*{hocolim}_{\mathcal{P}(s)_{<s}} * \simeq |\mathcal{P}(s)_{<s}| \simeq *$$

because $\mathcal{P}(s)_{<s}$ has the empty set as a minimum.

Finally we can prove that $|\mathcal{T}(d)^r_0|$ is $\Sigma_r$-equivalent to a point for all $r \geq 1$ (and $d > 2$). Fix some $H \leq \Sigma_r$. Since $\Sigma_r$ acts by permuting the factors of $|\mathcal{T}(d)^r|$, the simplices of $|\mathcal{T}(d)^r|^H$ are the $r$-tuples $(x_1,\dots,x_r)$ of simplices in $|\mathcal{T}(d)|$ such that $x_i = x_j$ if and only if $i$ and $j$ are in the same orbit of $H$. Thus, $|\mathcal{T}(d)^r|^H \cong |\mathcal{T}(d)^s|$ where $s$ is the number of orbits of $H$. Taking the intersection with $|\mathcal{T}(d)^r_0|$ we see that $|\mathcal{T}(d)^r_0|^H \cong |\mathcal{T}(d)^s_0|$ which we have shown to be contractible. Since this holds for any $H \leq \Sigma_r$, the map $|\mathcal{T}(d)^r_0| \to *$ is a $\Sigma_r$-equivalence.

**Case II: $d = 2$.** The argument above cannot be used because $|\mathcal{T}_0(2)| \not\simeq *$; it is the union of two points. We will show directly that $C^*(\mathcal{P}art(2)^r_0; \kappa_L^{\otimes \Sigma_r})^{\Sigma_r}$ has vanishing homology for $* \geq 1$. Let $\mathrm{Const}_L\colon \mathcal{P}art(d) \to \mathbf{Ab}$ be the constant functor. Since $\kappa_L(\lambda^{\min}) = L$ we obtain a short exact sequence of functors

$$0 \to \mathrm{Const}_L \to \kappa_L \to F \to 0.$$

Set $\mathcal{D} = \mathcal{P}art(d)^r_0$. By Proposition C.3 it remains to show that the cochain complex $E^*_F := C^*(\mathcal{D}; F^{\otimes r})^{\Sigma_r}$ is acyclic and that the homology of $E^*_L := C^*(\mathcal{D}; \mathrm{Const}_L)^{\Sigma_r}$ vanishes in positive degrees. We do this in Claims 1 and 2 below.

**Claim 1.** The homology of $E^*_L$ vanishes in positive degrees and $H^0(E^*_L) \cong L \le E^0_L = \prod_{[U]\subseteq \mathcal{P}art(2)^r_0} (L^r)^{\mathrm{Iso}(U)}$ via the inclusions $L = (L^r)^{\Sigma_r} \le (L^r)^{\mathrm{Iso}(U)}$.

*Proof.* By Observation C.8, $E^*_L = C^*_{\Sigma_r}(|\mathcal{P}art(2)^r_0|; \mathcal{H}^0(M))$ and $|\mathcal{P}art(2)^r_0| \to \mathrm{pt}$ is a $\Sigma_r$-equivalence because $\mathcal{P}art(2)^r_0$ contains a maximum $(\lambda^{\max}, \dots, \lambda^{\max})$. **q.e.d.**

**Claim 2.** If $d = 2$ then $E^*_F$ is acyclic.

*Proof.* Since $\mathcal{P}art(2)$ is isomorphic to the poset $\{0 < 1\}$, we can identify $\mathcal{P}art(2)^r_0$ with the poset $\mathcal{P}_0(r)$ of the non-empty subsets $U$ of $[r] = \{1, \dots, r\}$. With these identifications note that $F(0) = 0$ and $F(1) = L$ and we obtain a short exact sequence

$$0 \to F \to \mathrm{Const}_L \to G \to 0.$$

Proposition C.3 and Claim 1 imply that it remains to show that $E^*_L \to E^*_G$, where $E^*_G = C^*(\mathcal{P}(r); G^{\otimes r})^{\Sigma_r}$ induces an isomorphism in homology. Observe that if $U_\bullet$ is an $n$-simplex in $|\mathcal{P}(r)|$ of the form $U_0 \subseteq \cdots \subseteq U_n$ then $\mathrm{Iso}(U_\bullet) = \mathrm{Sym}(U^c_n) \times R$ where $U^c_n$ is the complement of $U_n$ in $[r]$ and $R \le \mathrm{Sym}(U_n)$. Therefore $\mathrm{Const}^{\otimes r}_L([U_\bullet])^{\Sigma_r} = (L^{U^c_n})^{\Sigma_{U^c_n}} \times (L^{U_n})^R$ and $(G^{\otimes r})([U_\bullet])^{\Sigma_r} = (L^{U^c_n})^{\mathrm{Sym}(U^c_n)}$ and the map between them is induced by the projection. Now, $G^{\otimes r}([U_\bullet])^{\Sigma_r} = L$ if $U_n \ne [r]$ and it vanishes if $U_n = [r]$. Therefore, by Proposition 2.3

$$E^*_G = C^*_{\Sigma_r}(|\mathcal{P}(r)_{<r}|; \mathrm{Const}_L) \cong C^*(\Delta^{r-2}; L)$$

where $\mathcal{P}_0(r)_{<r} = \mathcal{P}_0(r) - \{[r]\}$ and $\Delta^n$ is the standard $n$-simplex. Therefore $H(E^*_G)$ vanishes for $* \ge 1$ and $H^0(E^*_G) = L \le \prod_{[U]\subseteq \mathcal{P}_0(r)_{<r}} (L^{U^c})^{\mathrm{Sym}(U^c)}$ as the diagonal subgroup. By Claim 1 we see that $E^0_L \to E^0_G$ carries $Z^0(E^*_L) \cong L$ isomorphically onto $Z^0(E^*_G) \cong L$ and therefore $E^*_F$ is acyclic. **q.e.d.** □

## D. Proof of Propositions 7.2 and 7.1.

Throughout this section, $S^r_p(G)$ denotes the collection of the $p$-radical $p$-subgroups of a finite group $G$. See Definition 3.2. The subcollection of the $p$-centric $p$-radical subgroups is denoted $S^{rc}_p(G)$. If $N \lhd G$ then $S_p(N)$ is a $G$-poset via conjugation. In fact, $S^r_p(N)$ and $S^{rc}_p(N)$ are $G$-subposets.

**Proposition D.1** ([24, Proposition 4])**.** *Let $G$ be a finite group and $H \lhd G$. Then*

(1) *If $P$ is a $p$-radical $p$-subgroup of $G$ then $P \cap H$ is $p$-radical in $H$.*
(2) *If $P$ is a $p$-radical $p$-subgroup of $H$ then $\tilde{P} = O_p(N_G(P))$ is a $p$-radical subgroup of $G$ and $P = \tilde{P} \cap H$.*

This propositions enable us to make the following definition.

**Definition D.2.** Fix $p = 2$ and an even integer $n \geq 0$. Define the following functions $\alpha\colon S_p^r(\Sigma_n) \to S_p^r(A_n)$ and $\sigma\colon S_p^r(A_n) \to S_p^r(\Sigma_n)$ by the assignments

$$\alpha\colon P \mapsto P \cap A_n, \qquad \sigma\colon Q \mapsto O_p(N_{\Sigma_n}(Q)).$$

We will often use the notation $\tilde{Q}$ for $\sigma(Q)$.

Here are the key properties of $\alpha$ and $\sigma$. We will prove them later.

**Proposition D.3.** *Fix $p = 2$ and an even integer $n$. Then $\sigma(\alpha(P)) \leq P$ for all $P \in S_p^r(\Sigma_n)$.*

**Proposition D.4.** *The functions $\alpha$ and $\sigma$ defined above are $\Sigma_n$-equivariant and order preserving maps.*

**Proposition D.5.** *The maps $\alpha$ and $\sigma$ restrict to $\Sigma_n$-equivariant maps of posets.*

$$\alpha^c\colon S_p^{rc}(\Sigma_n) \to S_p^{rc}(A_n) \qquad \textit{and} \qquad \sigma^c\colon S_p^{rc}(A_n) \to S_p^{rc}(\Sigma_n).$$

*Moreover, if $P \in S_p^{rc}(\Sigma_n)$ then $C_{\Sigma_n}(\alpha(P)) \leq P$, hence $\alpha(P)$ is centric in $A_n$.*

***Proof of Proposition 7.1.*** By Proposition D.1(2) $\alpha^c \circ \sigma^c = \mathrm{id}_{S_p^{rc}(A_n)}$ so $\alpha^c$ is surjective and we now apply the last statement of Proposition D.5. □

***Proof of Proposition 7.2.*** By Proposition D.1(2), $\alpha \circ \sigma$ and $\alpha^c \circ \sigma^c$ are the identity functions on $S_p^r(A_n)$ and $S_p^{rc}(A_n)$. Proposition D.3 shows that inclusion of subgroups gives rise to natural transformations $\sigma \circ \alpha \to \mathrm{id}_{S_p^r(\Sigma_n)}$ and $\sigma^c \circ \alpha^c \to \mathrm{id}_{S_p^{rc}(\Sigma_n)}$. Therefore $|\sigma| \circ |\alpha|$ and $|\sigma^c| \circ |\alpha^c|$ are $\Sigma_n$-equivariantly homotopic to the identity on $|S_p^r(\Sigma_n)|$ and $|S_p^{rc}(\Sigma_n)|$. □

In the remainder of this section we will prove Propositions D.3-D.5.

**Proposition D.6** ([24, Prop. 1])**.** *Let $P$ be a $p$-radical $p$-subgroup of a finite group $G$. If $N_G(P)$ normalises a $p$-subgroup $Q \leq G$, then $Q \leq O_p(N_G(P)) = P$.*

***Proof of Proposition D.3.*** Clearly $N_{\Sigma_n}(P) \leq N_{\Sigma_n}(P \cap A_n) \leq N_{\Sigma_n}(\sigma \circ \alpha(P))$ since $A_n \lhd \Sigma_n$ and $\sigma \circ \alpha(P)$ is a characteristic subgroup of $N_{\Sigma_n}(P \cap A_n)$. Thus $\sigma(\alpha(P))$ is normalised by $N_{\Sigma_n}(P)$ and the result follows from Proposition D.6. □

**Definition D.7.** A basic subgroup $V_{e_1,\dots,e_r}$, see Definition 4.1, is of type I if $e_1 = 1$ and it is of type II if $e_1 \geq 2$.

**Lemma D.8.** *Fix $p = 2$. A basic $p$-subgroup of $\Sigma_n$ is contained in $A_n$ if and only if it is of type II.*

*Proof.* Fix $P = V_{c_1,\dots,c_r} = V_{c_1} \wr V_{c_2} \wr \cdots \wr V_{c_r}$. Every $V_{c_i}$ where $i \geq 2$ acts by permuting the orbits of $V_{c_1,\dots,c_{i-1}}$ whose length is $2^{c_1+\cdots+c_{i-1}}$, namely their length is even. Therefore $V_{c_i} \leq A_n$. Every $u \in V_{c_1} \cong C_2^{c_1}$ acts without fixed points on $2^{c_1}$ points and it is therefore a product of $2^{c_1-1}$ transpositions. Thus, $u \in A_n$ if and only if $c_1 \geq 2$. It follows that $P \leq A_n$ if and only if $c_1 \geq 2$. □

**Proposition D.9.** *Fix $p = 2$ and consider a product of basic subgroups of type II $P = \prod_{i=1}^t {V_{\mathbf{e}_i}}^{m_i}$ such that $|\operatorname{supp}(P)| = n$. Then $O_p(N_{\Sigma_n}(P)) \leq A_n$.*

*Proof.* Recall from Proposition 5.5 that $N(P)/P = \prod_i (\mathrm{GL}_{\mathbf{e}_i}(p) \wr \Sigma_{m_i})$. Now, $O_p(\mathrm{GL}_r(p)) = 1$ for any $r \geq 1$ because $\mathrm{GL}_1(2) = 1, \mathrm{GL}_2(2) = \Sigma_3$ and $\mathrm{GL}_r(2)$ is simple for $r \geq 3$. By Lemmas A.2 and D.8, $O_p(N(P))$ is generated by $P \leq A_n$ and by $O_p(\Sigma_{m_i})$ which are also contained in $A_n$ because each $\Sigma_{m_i}$ acts by permuting the orbits of $(V_{\mathbf{e}_i})^{m_i}$ whose cardinalities are $2^{d_i}$ where $d_i = \deg(V_{\mathbf{e}_i})$. □

**Lemma D.10.** *Fix $p = 2$ and a basic subgroup $P = V_{c_1,\dots,c_r} \leq \Sigma_n$. Then with the notation of Proposition 4.5, $R(\mathbf{c}) \leq A_n$ unless $P = V_2$.*

*Proof.* For every $1 \leq i \leq r$ set $f_i = c_1 + \cdots + c_i$. Observe that $\mathrm{GL}_{c_i}(p)$ acts diagonally on the base group $(V_{c_1,\dots,c_i})^{2^{d-f_i}}$ of $V_{\mathbf{c}} = V_{c_1,\dots,c_i} \wr V_{c_{i+1},\dots,c_r}$.

Assume first that $r \geq 2$. If $i < r$ then $d - f_i > 0$ hence $\mathrm{GL}_{c_i}(2)$ acts diagonally on the $2^{d-f_i}$ factors of the base group, so it must consist of even permutations. If $i = r$ then $\mathrm{GL}_{c_r}(2)$ acts by permuting the orbits of $(V_{c_1,\dots,c_{r-1}})^{2^{c_r}}$ so it also consists of even permutations.

If $r = 1$ then $\mathrm{GL}_{c_1}(p)$ is trivial if $c_1 = 1$ and it is simple if $c_1 \geq 3$ whence it is contained in $A_n$. If $c_1 = 2$ then $R(\mathbf{c}) = \Sigma_3$ embedded in the standard way in $\Sigma_n$. □

**Proposition D.11.** *Fix $p = 2$ and an even integer $n \geq 0$. If $Q \in S_p^r(A_n)$ then $\sigma(Q) = Q \cdot \delta_p(\sigma(Q))$; See Definition 4.7.*

*Proof.* Set $\tilde{Q} = \sigma(Q)$. Clearly $\tilde{Q} \supseteq Q \cdot \delta_p(\tilde{Q})$. From Proposition D.1 it follows that $\tilde{Q}$ is $p$-radical in $\Sigma_n$ and that $|\tilde{Q} \colon Q| \leq 2$. By [1, Section (2A)] $\tilde{Q}$ is a product of basic subgroups and $\delta_p(\tilde{Q}) = V_1^e$ by Lemma 4.9. If $\delta_p(\tilde{Q}) \neq 1$ then it contains odd permutations (Lemma D.8) and therefore $Q \lneq Q \cdot \delta_p(\tilde{Q})$ because $Q \leq A_n$, hence $\tilde{Q} = Q \cdot \delta_p(\tilde{Q})$. If $\delta_p(\tilde{Q}) = 1$ then Lemma 4.9 implies that $\tilde{Q}$ must be a product of basic subgroups of type II and therefore $\tilde{Q} \leq A_n$ by Lemma D.8. It now follows from Proposition D.1 that $Q = \tilde{Q} \cap A_n = \tilde{Q}$ whence $\tilde{Q} = Q \cdot \delta_p(\tilde{Q})$. □

***Proof of Proposition D.4.*** The assertions about $\alpha$ are immediate from the definitions since $A_n \lhd \Sigma_n$. The fact that $\sigma$ is $\Sigma_n$-equivariant is clear since $\Sigma_n$ acts on $S_p^r(A_n)$ by conjugation. It remains to prove that if $Q \leq P$ in $S_p^r(A_n)$ then $\tilde{Q} \leq \tilde{P}$.

By Proposition D.11 it suffices to show that $\delta_p(\tilde{Q}) \leq \delta_p(\tilde{P})$. Now, $\tilde{Q}$ and $\tilde{P}$ are products of basic subgroups because they are $p$-radical in $\Sigma_n$. Let $\tilde{Q}^{(\mathrm{II})}$ denote the product of the components of $\tilde{Q}$ of type II, see Definitions 5.3 and D.7. Similarly $\tilde{Q}^{(\mathrm{I})}$ is the product of the components of $\tilde{Q}$ of type I. By Lemma 4.9, $\delta_p(\tilde{Q}) = \delta_p(\tilde{Q}^{(\mathrm{I})}) = (V_1)^e$ for some $e \geq 0$. Note that $\operatorname{supp}(\tilde{Q}^{(\mathrm{I})}) = \operatorname{supp}(V_1^e)$. We examine the values of $e$.

**Case I.** If $e = 0$ then $\delta_p(\tilde{Q}) = 1 \leq \delta(\tilde{P})$ holds trivially.

**Case II.** Suppose that $e = 1$, namely $\delta_p(\tilde{Q}) = V_1$ generated by a transposition $(ij) \in \Sigma_n$. Then $\tilde{Q} = \tilde{Q}^{(\mathrm{II})} \times V_1$ and by Proposition D.1 and Lemma D.8, $Q = \tilde{Q}^{(\mathrm{II})}$ and in particular $Q$ fixes $m \geq 2$ points. Thus, $N_{\Sigma_n}(Q) = N_{\Sigma_{n-m}}(\tilde{Q}^{(\mathrm{II})}) \times \Sigma_m$. Therefore $\tilde{Q}^{(\mathrm{II})} \times V_1 = O_p(N_{\Sigma_n}(Q)) = \tilde{Q}^{(\mathrm{II})} \times O_p(\Sigma_m)$, hence $m = 2$ and $\tilde{Q} = \tilde{Q}^{(\mathrm{II})} \times \Sigma_2$.

Since $|\operatorname{supp}(\tilde{P})|$ is even and since $\tilde{P}$ contains $Q = \tilde{Q}^{(\mathrm{II})}$, it either has two fixed points $\{i, j\}$ or it has none. If it has two fixed points then $N_{\Sigma_n}(P) = N_{\Sigma_{n-2}}(P) \times \Sigma_2$ whence $\tilde{P} = O_p(N_{\Sigma_{n-2}}(P)) \times \Sigma_2$ so $\delta_p(\tilde{Q}) = \Sigma_2 \leq \delta_p(\tilde{P})$. Now assume that $\tilde{P}$ has no fixed points and present it as a product of basic subgroups $\tilde{P}_1 \times \cdots \times \tilde{P}_t$. Since $\{i, j\}$ are the only fixed points of $\tilde{Q}$ and every $\tilde{P}_k$ fixes an even number of points, we may assume that $\{i, j\} \subseteq \operatorname{supp}(\tilde{P}_1)$. Since $\tilde{P}_1 = V_{e_1} \wr V_{e_2,\dots,e_r}$, Lemma A.1 implies that $\tilde{P}_1$ is of type I and that $\{i, j\}$ is an orbit of $\delta_p(\tilde{P}_1) = (V_1)^\ell$. We deduce that $\delta_p(\tilde{P}) \geq \delta_p(\tilde{P}_1) \supseteq \operatorname{Sym}(\{i, j\}) = \delta_p(\tilde{Q})$.

**Case III.** $e = 2$, namely $\delta_p(\tilde{Q}) = (V_1)^2$. Then either $\tilde{Q} = \tilde{Q}^{(\mathrm{II})} \times (V_1)^2$ or $\tilde{Q} = \tilde{Q}^{(\mathrm{II})} \times V_{1,1}$ where $\tilde{Q}$ fixes $m$ elements. The former case is not possible because $N_{\Sigma_4}(V_1 \times V_1) = V_{1,1}$ so $\tilde{Q}$ is not radical. Therefore, $\tilde{Q} = \tilde{Q}^{(\mathrm{II})} \times V_{1,1}$ whence $Q = \tilde{Q} \cap A_n = \tilde{Q}^{(\mathrm{II})} \times V_2$ and therefore $N_{\Sigma_n}(Q) = N_{\Sigma_{n-m}}(\tilde{Q}^{(\mathrm{II})} \times V_2) \times \Sigma_m$. Since $\tilde{Q}$ fixes $m$ points, $O_p(\Sigma_m) = 1$ and $\tilde{Q} = O_p(N_{\Sigma_{n-m}}(\tilde{Q}^{(\mathrm{II})} \times V_2))$ which is contained in $A_{n-m}$ by Proposition D.9. Therefore $\delta_p(\tilde{Q}) = 1$ by Lemmas 4.9 and D.8. This is a contradiction showing that the case $e = 2$ is not possible.

**Case IV.** $\delta_p(\tilde{Q}) = (V_1)^e$ where $e \geq 3$. Note that $\delta_p(\tilde{Q}) \cap A_n$ has the same orbits as $\delta_p(\tilde{Q})$; there are $e$ orbits of length 2 and $n - 2e$ fixed points. Our goal is now to prove that if $\{\omega_1, \omega_2\}$ is an orbit of $\delta_p(\tilde{Q})$ then it is also an orbit of $\delta_p(\tilde{P}) = (V_1)^{e'}$. This will show that $\delta_p(\tilde{Q}) \leq \delta_p(\tilde{P})$.

Present $\tilde{P}$ as a product $\tilde{P}_1 \times \cdots \times \tilde{P}_t$ of basic subgroups. Since $\delta_p(\tilde{Q}) \cap A_n \leq Q \leq \tilde{P}$ (Proposition D.1), we may assume, without loss of generality, that $\{\omega_1, \omega_2\} \subseteq \operatorname{supp}(\tilde{P}_1)$. We will now show that $\delta_p(\tilde{P}_1)$ contains the transposition $(\omega_1 \omega_2)$.

Assume first that there is an orbit $\{\lambda_1, \lambda_2\}$ of $\delta_p(\tilde{Q})$ which is not contained in the support of $\tilde{P}_1$. Since $(\omega_1\omega_2)(\lambda_1\lambda_2) \in \delta_p(\tilde{Q}) \cap A_n \leq Q \leq \tilde{P}$ and since the supports of $\tilde{P}_1, \dots, \tilde{P}_t$ are disjoint, we deduce that $(\omega_1\omega_2) \in \tilde{P}_1$.

It remains to consider the case that $\operatorname{supp}(\tilde{P}_1)$ contains $\operatorname{supp}(\delta_p(\tilde{Q}) \cap A_n)$. Since $e \geq 3$ we deduce that $\delta_p(\tilde{Q}) \cap A_n$ contains a subgroup $H$ generated by $(\omega_1\omega_1)(\tau_1\tau_2)$ and $(\omega_1\omega_1)(\kappa_1\kappa_2)$ corresponding to 3 orbits of $\delta_p(\tilde{Q})$. Note that $H \leq Q$ by Proposition D.1. Say that $\tilde{P}_1 = V_{e_1,\dots,e_r}$ is basic of degree $d$. Since $1 \neq H \leq \delta_4(\tilde{P}_1)$, we deduce from Lemma 4.9 that $e_1 \leq 2$. In fact, by the same Lemma, $e_1 = 2$ is excluded because every element of $V_2$ supports 4 elements so $|\operatorname{supp}(H)|$ must be divisible by 4. But $H$ supports 6 elements, which is absurd. Therefore $e_1 = 1$.

Now, $\tilde{P}_1 = V_{e_1,\dots,e_r}$ where $e_1 = 1$. If $r = 1$ then $\tilde{P}_1 = V_1$ and it must support $\{\omega_1, \omega_2\}$ so $(\omega_1\omega_2) \in \tilde{P}_1$ as needed. If $r \geq 2$ we examine $e_2$. If $e_2 \geq 2$ then by Lemma A.1 (applied to $V_{e_1} \wr V_{e_2,\dots,e_r}$ and to $V_{e_2} \wr V_{e_3,\dots,e_r}$), the support of any element outside the base group $\delta_2(\tilde{P}_1)$ of $\tilde{P}_1$ contains at least $2^{e_1}2^{e_2} \geq 8$

elements whence $(\omega_1\omega_2)(\tau_1\tau_2) \in \delta_4(\tilde{P}_1) = \delta_2(\tilde{P}_1) = (V_1)^{p^{d-1}}$ which implies that $(\omega_1\omega_2) \in \delta_p(\tilde{P}_1)$. It remains to check the case $e_2 = 1$. By Lemma 4.9, $\delta_4(\tilde{P}_1) = (V_{1,1})^{p^{d-2}}$ and by its definition it contains $H$. Since $|\operatorname{supp}(V_{1,1})| = 4$, the support of either $(\omega_1\omega_2)(\tau_1\tau_2)$ or $(\omega_1\omega_2)(\kappa_1\kappa_2)$ must intersect two of the orbits of $\delta_4(\tilde{P}_1)$. Therefore $(\omega_1\omega_2)$ belongs to a factor of $(V_{1,1})^{p^{d-2}}$ and therefore it belongs to $\delta_2(\tilde{P}_1)$. □

***Proof of Proposition D.5***. **Step 1.** We first prove that $\sigma^c$ is well defined, namely, that if $Q \in S_p^{rc}(A_n)$ then $\tilde{Q} = \sigma(Q)$ is $p$-centric in $\Sigma_n$. Once $\sigma^c$ is well defined, it is clearly $\Sigma_n$-equivariant because $\sigma$ is by Proposition D.4.

Assume that $Q$ fixes $m$ points, thus $Q$ is a subgroup of $\Sigma_{n-m}$ and clearly $N_{\Sigma_n}(Q) = N_{\Sigma_{n-m}}(Q) \times \Sigma_m$ and $C_{\Sigma_n}(Q) = C_{\Sigma_{n-m}}(Q) \times \Sigma_m$. Therefore $C_{A_n}(Q)$ contains $A_m$ and since $Q$ is $p$-centric ($p = 2$), it follow that $A_m$ must be trivial, hence $m \leq 2$. Since $n$ is even and $Q$ is a 2-group, $m$ is even, so $m = 2$ or $m = 0$. If $m = 2$ then $\tilde{Q} = O_p(N_{\Sigma_n}(Q))$ contains $Q \times \Sigma_2$ and therefore $\tilde{Q}$ has no fixed points. If $m = 0$ then $\tilde{Q}$ has no fixed points because it contains $Q$. Since $\tilde{Q}$ is $p$-radical, hence a product of basic subgroups, it follows from Proposition 5.5 that $\tilde{Q}$ is $p$-centric.

**Step 2.** We prove that $\alpha^c$ is well defined, namely, that if $P \in S_p^{rc}(\Sigma_n)$ then $\alpha(P)$ is $p$-centric in $A_n$. In fact we will prove that $C_{\Sigma_n}(P \cap A_n) \leq P$, whence $C_{A_n}(P \cap A_n) \leq P \cap A_n$. This will also prove the last statement of this proposition because once $\sigma^c$ and $\alpha^c$ are well defined, then Proposition D.1(2) implies that $\alpha^c$ is surjective onto $S_p^{rc}(A_n)$.

Present $P$ as a product of basic subgroups and let $P^{(\mathrm{II})}$ be the product of the components of type II and let $P^{(\mathrm{I})}$ denote the product of the components of type I, see Definition D.7. Let $S$ and $T$ be the supports of $P^{(\mathrm{II})}$ and $P^{(\mathrm{I})}$ respectively and set $s = |S|$ and $t = |T|$. Since $P$ is $p$-centric, Proposition 5.5 implies that $S$ and $T$ form a partition of $\Omega$ where we identify $\Sigma_n$ with $\operatorname{Sym}(\Omega)$. Note that $P^{(\mathrm{II})} \leq A_s$ by Lemma D.8 and that $N_{\Sigma_n}(P) = N_{\Sigma_s}(P^{(\mathrm{II})}) \times N_{\Sigma_t}(P^{(\mathrm{I})})$ by Proposition 5.5. Thus, $P^{(\mathrm{II})}$ and $P^{(\mathrm{I})}$ are $p$-radical in $\Sigma_s$ and $\Sigma_t$.

By Lemma 4.9 we have $\delta_p(P^{(\mathrm{I})}) = (V_1)^e$. If $e = 0$ then $P^{(\mathrm{I})} = 1$, namely $P = P^{(\mathrm{II})}$ which by Lemma D.8 implies that $P \leq A_n$. Hence, if $e = 0$ then $C_{\Sigma_n}(P \cap A_n) = C_{\Sigma_n}(P) \leq P$ by Proposition 5.5.

If $e = 1$ then $P^{(\mathrm{I})} = V_1$ and $s = n - 2$ and $t = 2$. Since $P^{(\mathrm{II})} \leq A_{n-2}$ by Lemma D.8, $P \cap A_n = (P^{(\mathrm{II})} \times V_1) \cap A_n = P^{(\mathrm{II})}$ and therefore by Propositions A.3(1) and 5.5, $C_{\Sigma_n}(P \cap A_n) = C_{\Sigma_{n-2}}(P^{(\mathrm{II})}) \times \Sigma_2 = Z(P^{(\mathrm{II})}) \times V_1 \leq P$ as needed.

Assume that $e \geq 2$ and notice that $\operatorname{supp}(P^{(\mathrm{I})} \cap A_t) = \operatorname{supp}(P^{(\mathrm{I})})$ because

$$\operatorname{supp}(P^{(\mathrm{I})}) = \operatorname{supp}((V_1)^e) = \operatorname{supp}((V_1)^e \cap A_t) \subseteq \operatorname{supp}(P^{(\mathrm{I})} \cap A_t) \subseteq \operatorname{supp}(P^{(\mathrm{I})}).$$

By Proposition A.3, $C_{\Sigma_n}(P \cap A_n) = C_{\Sigma_n}(P^{(\mathrm{II})} \times (P^{(\mathrm{I})} \cap A_t)) = C_{\Sigma_s}(P^{(\mathrm{II})}) \times C_{\Sigma_t}(P^{(\mathrm{I})} \cap A_t)$. Since $C_{\Sigma_s}(P^{(\mathrm{II})}) \leq P^{(\mathrm{II})}$ by Proposition 5.5, it remains to prove that

$$C_{\Sigma_t}(P^{(\mathrm{I})} \cap A_t) \leq P^{(\mathrm{I})}.$$

Note that $t \geq 4$ because $e \geq 2$. Present $P^{(\mathrm{I})} = P_1 \times \cdots \times P_k$ as a product of basic subgroups of type I in $\Sigma_t$.

If $k \geq 3$, we deduce from Lemma D.8, Proposition A.3(2) and from Proposition 5.5 that $C_{\Sigma_t}(P^{(\mathrm{I})} \cap A_t) = C_{\Sigma_t}(P^{(\mathrm{I})}) \leq P^{(\mathrm{I})}$.

If $k = 2$, namely if $P = P_1 \times P_2$ where $\deg(P_2) \leq \deg(P_1) = d$ then $|\operatorname{supp}(P_1)| = 2^d$. If $d \geq 2$ then Lemma D.8, Proposition A.3(3) and Proposition 5.5 imply $C_{\Sigma_t}(P^{(\mathrm{I})} \cap A_t) = C_{\Sigma_t}(P^{(\mathrm{I})}) \leq P^{(\mathrm{I})}$. If $d = 1$ then $t = 4$ and $P^{(\mathrm{I})} = V_1 \times V_1$ and we observe that $N_{\Sigma_4}(V_1 \times V_1) = V_{1,1}$ contradicting the fact, which we have seen above, that $P^{(\mathrm{I})}$ is $p$-radical in $\Sigma_4$. So $d = 1$ is impossible.

Finally, if $k = 1$, namely $P^{(\mathrm{I})}$ is a basic subgroup of type I and degree $d$, then by Lemma 4.9

$$P^{(\mathrm{I})} \cap A_t \supseteq \delta_p(P^{(\mathrm{I})}) \cap A_t = (V_1)^{2^{d-1}} \cap A_t.$$

Note that $d \geq 2$ because $e \geq 2$ so $|\operatorname{supp}(P^{(\mathrm{I})})| \geq 2^e \geq 4$. If $d > 2$ then by Proposition A.3(2), $C_{\Sigma_t}(P^{(\mathrm{I})} \cap A_t) \leq C_{\Sigma_t}((V_1)^{2^{d-1}} \cap A_t) = C_{\Sigma_t}((V_1)^{2^{d-1}}) = (V_1)^{2^{d-1}} \leq P^{(\mathrm{I})}$. If $d = 2$ then $t = 4$ and $P^{(\mathrm{I})} = V_{1,1}$ because $V_{1,1}$ is the only basic subgroup of degree 2 and type I. Therefore $P^{(\mathrm{I})} \cap A_t = V_2$ and we observe that $C_{\Sigma_4}(V_2) = V_2 \leq P^{(\mathrm{I})}$. $\square$

## E. Proof of the results in section 4

.

**Lemma E.1.** *Let $P = V_{c_1,\dots,c_t}$ be a basic subgroup of degree $d$ in $\Sigma_{p^d}$. If $Q \leq P$ is a basic subgroup of degree $d$ which is conjugate to some $V_{m_1,\dots,m_s}$ then*

*(a) The partition $(c_1, \dots, c_t)$ refines $(m_1, \dots, m_s)$, namely there are integers $q_1, \dots, q_{s-1}$ such that*

$$\underbrace{c_1 + \cdots + c_{q_1}}_{m_1}, \underbrace{c_{q_1+1} + \cdots + c_{q_2}}_{m_2}, \underbrace{c_{q_2+1} + \cdots + c_{q_3}}_{m_3}, \dots, \underbrace{c_{q_{s-1}+1} \cdots + c_t}_{m_s}$$

*(b) If $R \leq P$ is conjugate to $Q$ in $\Sigma_{p^d}$, then $R$ and $Q$ are $P$-conjugate.*

*Proof.* (a) Assume that there are $a < s$ and $q < t$ such that

$$c_1 + \cdots + c_q < m_1 + \cdots + m_a < c_1 + \cdots + c_{q+1}.$$

Set $r = p^{m_1 + \cdots + m_a}$ and set $e = p^{c_{q+1} + \cdots + c_t}$ and $f = p^{m_{a+1} + \cdots + m_s}$. Clearly $e > f$ because $\sum c_i = \sum m_i = d$. By Lemma 4.9, $\delta_r(Q) = (V_{m_1,\dots,m_a})^f$ and $\delta_r(P) = (V_{c_1,\dots,c_q})^e$. Since all basic groups act transitively, $\delta_r(Q)$ has $f$ orbits and $\delta_r(P)$ has $e$ orbits. But $\delta_r(Q) \leq \delta_r(P)$ and $e > f$ which is absurd.

(b) Set $n = p^d$. We prove the result by induction on $s$.

**Step 1:** $s = 1$. In this case $Q$ and $R$ are conjugate to $V_d$. We will prove that $Q$ is conjugate to $R$ in $P$ by induction on $t$. If $t = 1$ then $P = V_d$ so $Q = R = P$ and the result is a triviality. Fix some $t > 1$. Set $e = p^{c_2 + \cdots + c_t}$ and set $H = V_{c_1}$

and $K = V_{c_2,\dots,c_t} \leq \Sigma_e$. Then $P = H \wr K$ is a subgroup of $H \wr \Sigma_e$. Let $\Delta^e H$ denote the diagonal subgroup of $H^e$ and note that it is central in $P$. In fact $\Delta^e H = Z(P)$. Since $Q$ is abelian and acts freely and transitively and since $Q \leq P$,

$$Q = C_{\Sigma_n}(Q) \supseteq C_{\Sigma_n}(P) \supseteq \Delta^e H \cong H.$$

Set $Q' = Q \cap H^e$ and let $Q''$ be a complement for $Q'$ in $Q \cong V_d$. Clearly, $Q' \supseteq \Delta^e H$. Now, $Q$ acts freely, namely with trivial isotropy groups, and therefore so does $Q'$. Since $Q' \leq H^e$, the length of its orbits is at most $|H| = p^{c_1}$. Since $Q'$ acts freely $|Q'| \leq p^{c_1} = |\Delta^e H|$ and therefore $Q' = \Delta^e H$.

Similarly, set $R' = R \cap H^e$ and let $R''$ be a complement of $R'$ in $R$. By the argument above, $R' = \Delta^e H$. Note that $Q''$ and $R''$ are elementary abelian of order $|Q|/|Q'| = p^{d-c_1} = e$.

Let $\bar{Q}, \bar{R}$ and $\bar{P}$ be the images of $Q, R$ and $P$ under the projection $H \wr \Sigma_e \to \Sigma_e$. Clearly $\bar{P} = K$ and $Q''$ and $R''$ are mapped isomorphically onto $\bar{Q}$ and $\bar{R}$. Since $Q$ and $R$ are transitive, so are $\bar{Q}$ and $\bar{R}$, and since $|\bar{Q}| = |\bar{R}| = e$ they act freely. Thus, $\bar{Q}$ and $\bar{R}$ are conjugate in $\Sigma_e$ to $V_{d-c_1}$ and both are subgroups of $\bar{P}$. By the induction hypothesis on $t$ there is some $g \in \bar{P} \leq K \leq P$ which conjugates $\bar{Q}$ to $\bar{R}$ and we may therefore assume that $\bar{Q} = \bar{R}$ (note that $g$ centralises $Q' = R'$). By Proposition A.8 both $Q''$ and $R''$ are $H^e$-conjugate, and therefore $P$-conjugate to $\bar{Q} = \bar{R}$ and since $Q' = R' = \Delta^e H$ are fixed by this conjugation, we deduce that $Q$ is conjugate to $R$ in $P$. This completes the induction step for $t$.

**Step 2: The induction step for $s > 1$.** Set $r = p^{m_1+\cdots+m_{s-1}}$ and $e = p^{m_s}$. Set $Q' = \delta_r(Q), R' = \delta_r(R)$ and $P' = \delta_r(P)$. By part (a) and Lemma 4.9, $P' = (V_{c_1,\dots,c_q})^e$ for some $q < t$ and $Q'$ and $R'$ are conjugate to $(V_{m_1,\dots,m_{s-1}})^e$. In addition, $Q'$ and $R'$ have complements $Q'', R'' \cong V_{m_s}$ in $Q$ and $R$ respectively.

Note that $P = H \wr K$ where $H = V_{c_1,\dots,c_q}$ and $K = V_{c_{q+1},\dots,c_t}$. Since $Q', R' \leq P' = H^e$ and all these groups have exactly $e$ orbits of length $r$, one easily sees that every factor of $P'$ contains exactly one factor of $Q'$ and one factor of $R'$. Clearly, $H^e \cap Q \leq \delta_r(Q)$ and therefore $Q' = H^e \cap Q$. Similarly $R' = H^e \cap R$.

Let $\bar{Q}, \bar{R}$ and $\bar{P}$ be the images of $Q, R$ and $P$ under the projection $H \wr \Sigma_e \to \Sigma_e$. Then $\bar{P} = K$ and $\bar{Q}$ and $\bar{R}$ are elementary abelian groups of order $e = p^{m_s}$. They act transitively because $Q$ and $R$ are transitive and therefore they act freely and therefore they are conjugate to $V_{m_s}$. By the case $s = 1$ shown above, $\bar{Q}$ and $\bar{R}$ are $K$-conjugate and since $K \leq P$ we can assume $\bar{Q} = \bar{R}$. Also $Q''$ and $R''$ maps isomorphically onto $\bar{Q}$ and $\bar{R}$ because $Q' = H^e \cap Q$ and $R' = H^e \cap R$. By Proposition A.8, $Q''$ is $H^e$-conjugate to $\bar{Q}$, hence $P$-conjugate. Similarly, $R''$ is $P$-conjugate to $\bar{R}$ and we therefore assume that $Q'' = \bar{Q} = \bar{R} = R''$. Consider one factor $H = V_{c_1,\dots,c_q}$ of $P'$. We have seen that it contains a factor $Q'_{(1)}$ of $Q'$ and one factor $R'_{(1)}$ of $R'$, both are conjugate to $V_{m_1,\dots,m_{s-1}}$. By induction hypothesis on $s$, there is some $g \in H$ which conjugates $Q'_{(1)}$ to $R'_{(1)}$. Consider the element $\mathbf{g} = (g, \dots, g) \in H^e = P'$. It clearly conjugates $Q'_{(1)}$ to $R'_{(1)}$ and it also centralises $\Sigma_e \leq H \wr \Sigma_e$ and therefore it centralises $\bar{Q}$ and $\bar{R}$. Thus, after conjugation by $\mathbf{g}$ we get that $Q'' = \bar{Q} = \bar{R} = R''$ and $Q'_{(1)} = R'_{(1)}$. The result follows since $Q$ is generated by $Q''$ and by $Q'_{(1)}$ and $R$ is generated by $R''$ and by $R'_{(1)}$. $\square$

***Proof of Proposition 4.5.*** Let $\mathrm{Sym}(X)$ denote the symmetric group of a set $X$. If $Y \subseteq X$ then $\mathrm{Sym}(Y)$ is a subgroup of $\mathrm{Sym}(X)$ which fixes $X - Y$.

We view the elements of $\mathcal{P}art(d)$ as partitions into intervals of the standard basis of the vector space $U = \mathbb{F}_p^d$. We will identify $\Sigma_n$ with $\mathrm{Sym}(U)$.

Fix some $\mathbf{c} = (c_1, \dots, c_t)$ in $\mathcal{P}art(d)$ and let $W_j$ denote the subspace of $U$ generated by the basis elements in the $j$th interval; Thus $\dim W_j = c_j$. For every $1 \leq j \leq t$ set $F_j = \sum_{i=1}^{j} W_i$ and let $W_j$ act on $F_j$ via left translation. Now, $F_j$ is a subset of $U$ and let $\tilde{W}_j$ denote the image of the composite $W_j \to \mathrm{Sym}(F_j) \leq \mathrm{Sym}(U)$. It is clear that if $\tilde{w}$ is the image of $w \in W_j$ in $\tilde{W}_j$ and if $u \in U$ then $\tilde{w}(u) = w + u$ if $u \in F_j$ and $\tilde{w}(u) = u$ if $u \notin F_j$.

Since $F_j = W_j \oplus F_{j-1}$, we see that $\tilde{W}_j$ acts on $F_j \cong \coprod_{w \in W_j} F_{j-1}$ by permuting the components. Therefore, if $K$ is any subgroup of $\mathrm{Sym}(F_{j-1})$ then $\tilde{W}_j$ and $K$ generate $K \wr W_j$. We define

$$V(\mathbf{c}) \stackrel{\text{def}}{=} \text{the subgroup of } \mathrm{Sym}(U) \text{ generated by } \tilde{W}_1, \dots, \tilde{W}_t.$$

It is immediate from Definition 4.1 that $V(\mathbf{c})$ is the basic subgroup $V_{c_1,\dots,c_t}$. Since $W_{j-1}$ acts on $F_j = \coprod_{w \in W_j} F_{j-1}$ via its action on $F_{j-1}$, the image of

$$W_{j-1} \oplus W_j \to \mathrm{Sym}(F_j) \leq \mathrm{Sym}(U)$$

is generated by $\tilde{W}_j$ and the diagonal copy of $W_{j-1}$ in $W_{j-1} \wr W_j \cong \langle \tilde{W}_{j-1}, \tilde{W}_j \rangle$. In particular its image is contained in $\langle \tilde{W}_{j-1}, \tilde{W}_j \rangle$. It now follows that if $\mathbf{c}' \leq \mathbf{c}$ then $V(\mathbf{c}') \leq V(\mathbf{c})$ because the partition $\mathbf{c}'$ is obtained from $\mathbf{c}$ by successively joining adjacent intervals. This completes the proof of point (i). Point (ii) follows immediately from Lemma E.1 and we now prove point (iii).

For $\mathbf{c} \in \mathcal{P}art(d)$ of the form $(c_1, \dots, c_t)$ let $R(\mathbf{c})$ denote $\mathrm{GL}(W_1) \times \cdots \times \mathrm{GL}(W_t)$ as a subgroup of $\mathrm{GL}(U) \leq \mathrm{Sym}(U)$. It is clear that $R(\mathbf{c}') \leq R(\mathbf{c})$ if $\mathbf{c} \leq \mathbf{c}'$. If $g \in \mathrm{GL}(W_j) \subseteq R(\mathbf{c})$ then $g(W_i) = W_i$ and $g(F_i) = F_i$ for all $i$. If $w \in W_i$ then $g\tilde{w}g^{-1} = \widetilde{g(w)}$ because for every $u \in U$

$$g \circ \tilde{w} \circ g^{-1}(u) = \begin{cases} g(w) + u & \text{if } u \in F_i \\ u & \text{if } u \notin F_i \end{cases}$$

We deduce that $g$ normalises $\tilde{W}_j$ and centralises $\tilde{W}_i$ if $i \neq j$. Therefore $R(\mathbf{c})$ normalises $V(\mathbf{c}) = \langle \tilde{W}_1, \dots, \tilde{W}_t \rangle$. Finally, $R(\mathbf{c})$ acts on $V(\mathbf{c}) = W_{c_1} \wr \cdots \wr W_{c_t}$ via the action of $\mathrm{GL}(W_i)$ on $W_i$. It is not hard to show that $V(\mathbf{c}) \cap R(\mathbf{c}) = 1$ because $V_c$ acts freely so$V_c \cap \mathrm{GL}_c(p) = 1$ for any $c > 0$ and therefore $V(\mathbf{c}) = V_{c_1} \wr \cdots \wr V_{c_t}$ intersects $\mathrm{GL}_{c_1}(p) \wr \cdots \wr \mathrm{GL}_{c_t}(p) = 1$ trivially. It follows from [1, Section 2] that $N_{\Sigma_n}(V(\mathbf{c})) = V(\mathbf{c}) \rtimes R(\mathbf{c})$.

Fix $\mathbf{c} = (c_1, \dots, c_t)$ in $\mathcal{P}art(d)$. Every $\mathrm{GL}(W_j)$ contains the subgroup $D$ of the matrices $\mathrm{diag}(\mathbb{F}_p^\times, 1, \dots, 1)$. Clearly $D$ is isomorphic to $\mathbb{F}_p^\times$ which is cyclic of order $p-1$ and it acts on each one of the $p^{d-1}$ cosets of $\mathbb{Z}/p \subseteq W_j \subseteq U$ by fixing one point and cyclically permuting the rest. Therefore a generator of $D$ is a product of $p^{d-1}$ cycles of length $p-1$ which is an odd permutation since $p$ is odd. $\square$

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